GARCH models without positivity constraints: Exponential or Log GARCH ?*

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Objectives

Log-GARCH and EGARCH are two models for the log-volatility.

- Probabilistic properties and estimation of asymmetric Log-GARCH models.
- Differences and similarities between the log-GARCH and EGARCH models.
- Testing log-GARCH against EGARCH, or the reverse.

The standard GARCH model

Engle (1982), Bollerslev (1986)

Standard GARCH models:

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, \quad (\eta_t)_{t\in\mathbb{Z}} \text{iid}(0,1) \\ \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \end{cases}$$

with positivity constraints $\omega > 0, \ \alpha_i, \beta_j \geq 0$.

Under relevant conditions on the parameter, the model is able to mimic some properties of the financial returns:

- this is a conditionally heterosckedastic white noise;
- the squares are positively autocorrelated;
- the model generates volatility clustering;
- the marginal distribution can be leptokurtic.

Two drawbacks of the standard GARCH

1. Do not allows for asymmetries in volatility (leverage effects): decreases of prices have an higher impact on the future volatility than increases of the same magnitude.
   ▸ Leverage effects
2. The positivity constraints on the volatility coefficients entail numerical and statistical difficulties (*e.g.* non standard asymptotic distribution of constrained estimators at the boundary of the parameter space).

⇒ numerous extensions (see Bollerslev "Glossary to ARCH (GARCH)", 2009)

Two log-volatility models

$\epsilon_t = \sigma_t \eta_t$, η_t iid $(0,1)$

Exponential-GARCH model: Nelson (1991)

$$\log \sigma_t^2 = \omega + \sum_{j=1}^{p} \beta_j \log \sigma_{t-j}^2 + \sum_{i=1}^{\ell} \gamma_{i+} \eta_{t-i}^+ + \gamma_{i-} \eta_{t-i}^-$$

Asymmetric log-GARCH model:

$$\begin{aligned} \log \sigma_t^2 &= \omega + \sum_{j=1}^{p} \beta_j \log \sigma_{t-j}^2 \\ &+ \sum_{i=1}^{q} \left(\alpha_{i+} 1_{\{\epsilon_{t-i}>0\}} + \alpha_{i-} 1_{\{\epsilon_{t-i}<0\}} \right) \log \epsilon_{t-i}^2 \end{aligned}$$

The log-GARCH model has been introduced by Geweke (1986), Pantula (1986) and Milhøj (1987) (see Sucarrat and Escribano (2010) for the symmetric case).

Basic features of the log-GARCH model

Symmetric log-GARCH(1,1):

$$\begin{aligned}\log \sigma_t^2 &= \omega + \beta \log \sigma_{t-1}^2 + \alpha \log \epsilon_{t-1}^2 \\ &= \omega + (\alpha + \beta) \log \sigma_{t-1}^2 + \alpha \log \eta_{t-1}^2.\end{aligned}$$

Symmetric EGARCH(1,1):

$$\begin{aligned}\log \sigma_t^2 &= \omega + \beta \log \sigma_{t-1}^2 + \gamma |\eta_{t-1}| \\ &= \omega + \beta \log \sigma_{t-1}^2 + \gamma \left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right|.\end{aligned}$$

- No positivity constraint on the parameters, but $|\eta_t| > 0$.
- Easily invertible, contrary to the EGARCH.
- The volatility is not bounded below by a strictly positive constant.
- Small values have persistent effects on volatility.

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- No positivity constraint on the parameters, but $|\eta_t|>0$.
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Basic features of the asymmetric log-GARCH model

Asymmetric log-GARCH(1,1):

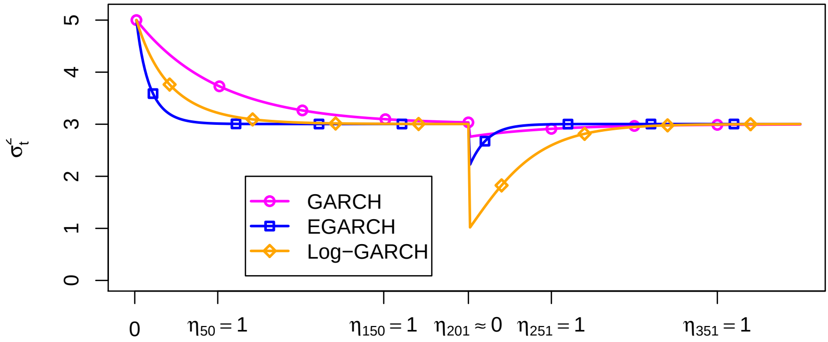
$$\begin{aligned}\log\sigma_t^2 &= \omega+\beta\log\sigma_{t-1}^2+\left(\alpha_+1_{\{\epsilon_{t-1}>0\}}+\alpha_-1_{\{\epsilon_{t-1}<0\}}\right)\log\epsilon_{t-1}^2\\ &= \omega+\left(\alpha_+1_{\{\eta_{t-1}>0\}}+\alpha_-1_{\{\eta_{t-1}<0\}}+\beta\right)\log\sigma_{t-1}^2\\ &\quad +\left(\alpha_+1_{\{\eta_{t-1}>0\}}+\alpha_-1_{\{\eta_{t-1}<0\}}\right)\log\eta_{t-1}^2.\end{aligned}$$

Asymmetric EGARCH(1,1):

$$\log\sigma_t^2 = \omega+\beta\log\sigma_{t-1}^2+\gamma_+\eta_{t-1}^+ +\gamma_-\eta_{t-1}^-.$$

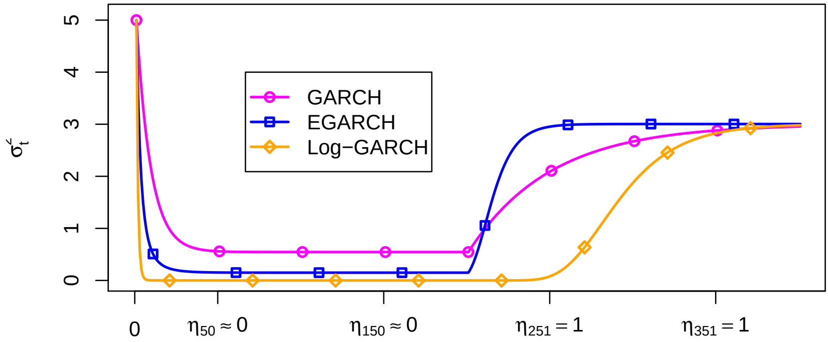
- Asymmetric random persistence parameter.

Effect of a small value



GARCH: $\sigma_t^2 = 0.06 + 0.09\epsilon_{t-1}^2 + 0.89\sigma_{t-1}^2$
log-GARCH: $\log\sigma_t^2 = 0.033 + 0.03\log\epsilon_{t-1}^2 + 0.93\log\sigma_{t-1}^2$
EGARCH: $\log\sigma_t^2 = 0.044 + 0.3|\eta_{t-1}| + 0.9\log\sigma_{t-1}^2$

Effect of a sequence of small values



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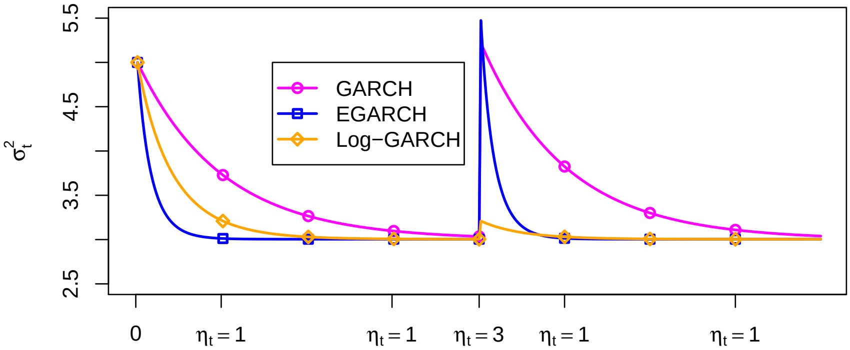

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Markovian representation

The log-GARCH model $\epsilon_t = \sigma_t \eta_t$ with

$$\log \sigma_t^2 = \omega + \sum_{i=1}^{q} \left(\alpha_{i+} 1_{\{\epsilon_{t-i}>0\}} + \alpha_{i-} 1_{\{\epsilon_{t-i}<0\}} \right) \log \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \log \sigma_{t-j}^2$$

can be written as

$$\boldsymbol{z}_t = \boldsymbol{C}_t \boldsymbol{z}_{t-1} + \boldsymbol{b}_t$$

where, in the case $(p, q) = (1, 1)$,

$$\boldsymbol{z}_t = \left(1_{\{\epsilon_t>0\}} \log \epsilon_t^2, \quad 1_{\{\epsilon_t<0\}} \log \epsilon_t^2, \quad \log \sigma_t^2 \right)',$$

$$\boldsymbol{b}_t = \left((\omega + \log \eta_t^2) 1_{\{\eta_t>0\}}, \quad (\omega + \log \eta_t^2) 1_{\{\eta_t<0\}}, \quad \omega \right)'$$

and

$$\boldsymbol{C}_t = \begin{pmatrix} 1_{\{\eta_t>0\}} \alpha_+ & 1_{\{\eta_t>0\}} \alpha_- & 1_{\{\eta_t>0\}} \beta \\ 1_{\{\eta_t<0\}} \alpha_+ & 1_{\{\eta_t<0\}} \alpha_- & 1_{\{\eta_t<0\}} \beta \\ \alpha_+ & \alpha_- & \beta \end{pmatrix}.$$

Strict stationarity

Sufficient condition:

Assume that $E\log^+|\log\eta_0^2| < \infty$. A sufficient condition for the existence of a (unique) strictly stationary (and non anticipative) solution to the log-GARCH model is $\gamma(\mathbf{C}) < 0$, with

$$\gamma(\mathbf{C}) = \lim_{t\to\infty} \frac{1}{t} E\left(\log \|\boldsymbol{C}_t \boldsymbol{C}_{t-1} \dots \boldsymbol{C}_1\|\right) = \inf_{t\geq 1} \frac{1}{t} E(\log \|\boldsymbol{C}_t \boldsymbol{C}_{t-1} \dots \boldsymbol{C}_1\|).$$

▸ The log-GARCH(1,1) case

Following Bougerol and Picard (1992), the condition is necessary only under an irreducibility assumption.

▸ Example

Another Markovian representation

The log-GARCH model $\epsilon_t = \sigma_t \eta_t$ with

$$\log \sigma_t^2 = \omega + \sum_{i=1}^{q} \left(\alpha_{i+} 1_{\{\epsilon_{t-i}>0\}} + \alpha_{i-} 1_{\{\epsilon_{t-i}<0\}} \right) \log \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \log \sigma_{t-j}^2$$

can be written as

$$\boldsymbol{\sigma}_{t,r} = \boldsymbol{A}_t \boldsymbol{\sigma}_{t-1,r} + \boldsymbol{u}_t,$$

where $r = \max\{p,q\}$, $\boldsymbol{\sigma}_{t,r} = (\log \sigma_t^2, \dots, \log \sigma_{t-r+1}^2)'$ and

$$\boldsymbol{A}_t = \begin{pmatrix} \mu_1(\eta_{t-1}) & \dots & \mu_{r-1}(\eta_{t-r+1}) & \mu_r(\eta_{t-r}) \\ & \boldsymbol{I}_{r-1} & & \boldsymbol{0}_{r-1} \end{pmatrix},$$

$$\mu_i(\eta_{t-i}) = \alpha_{i+} 1_{\{\eta_{t-i}>0\}} + \alpha_{i-} 1_{\{\eta_{t-i}<0\}} + \beta_i,$$

log-GARCH(1,1) case

Remark on the two Markovian representations

In $\boldsymbol{z}_t = \boldsymbol{C}_t \boldsymbol{z}_{t-1} + \boldsymbol{b}_t \in \mathbb{R}^{2q+p}$ the matrices $(\boldsymbol{C}_t)$ are iid.
In $\boldsymbol{\sigma}_{t,r} = \boldsymbol{A}_t \boldsymbol{\sigma}_{t-1,r} + \boldsymbol{u}_t \in \mathbb{R}^r$, $r = \max(p,q)$, the matrices

$$\boldsymbol{A}_t = \begin{pmatrix} \mu_1(\eta_{t-1}) & \dots & \mu_{r-1}(\eta_{t-r+1}) & \mu_r(\eta_{t-r}) \\ & \boldsymbol{I}_{r-1} & & \boldsymbol{0}_{r-1} \end{pmatrix}$$

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are not independent, but

☺ $$E \prod_{t=1}^{\ell} \boldsymbol{A}_t = \prod_{t=1}^{\ell} E\boldsymbol{A}_t.$$

Existence of a fractional log-moment

For standard GARCH, $\gamma < 0 \Rightarrow E|\epsilon_t|^s < \infty$ for some $s > 0$.

Also the case for $|\log \epsilon_t^2|$ in the log-GARCH model, if the condition $E \log^+ |\log \eta_0^2| < \infty$ is slightly reinforced.

Existence of some log-moment of order $s > 0$

Assume $\gamma(\mathbf{C}) < 0$ and $E|\log \eta_0^2|^{s_0} < \infty$ for some $s_0 > 0$:
$\exists s > 0$ such that $E|\log \epsilon_t^2|^s < \infty$ and $E|\log \sigma_t^2|^s < \infty$.

Existence of $E|\log \epsilon_t^2|$

Let $\boldsymbol{A}^{(m)} = E\{\mathrm{Abs}(\boldsymbol{A}_1)\}^{\otimes m}$ where $\mathrm{Abs}(\boldsymbol{A}) = (|\boldsymbol{A}_{ij}|)$.

Existence of a log-moment of order 1

Assume that $\gamma(\mathbf{C}) < 0$ and that $E|\log \eta_0^2| < \infty$. If

$$\rho(\boldsymbol{A}^{(1)}) < 1,$$

then

$$E|\log \epsilon_t^2| < \infty \quad \text{and} \quad E|\log \sigma_t^2| < \infty.$$

Similarly, log-moments of order m exist if

$$\gamma(\mathbf{C}) < 0, \quad E|\log \eta_0^2|^m < \infty, \quad \rho(\boldsymbol{C}^{(m)}) < 1.$$

Existence of $E|\log\epsilon_t^2|^m$ for all $m \in \mathbb{N}^*$

$\boldsymbol{A}^{(\infty)} = \text{ess}\sup \text{Abs}(\boldsymbol{A}_1)$ be the essential supremum of $\text{Abs}(\boldsymbol{A}_1)$ term by term.

Existence of log-moments of any order

Assume that $\gamma(\mathbf{C}) < 0$. If

$$\rho(\boldsymbol{A}^{(\infty)}) < 1 \Leftrightarrow \sum_{i=1}^{r} \max(|\alpha_{i+} + \beta_i|, |\alpha_{i-} + \beta_i|) < 1,$$

then

$$E|\log\epsilon_t^2|^m < \infty \quad \text{and} \quad E|\log\sigma_t^2|^m < \infty$$

for all m such that $E|\log\eta_0^2|^m < \infty$.

▸ Symmetric case

Existence of moments

Existence of moments of any order

Assume that $\gamma(\mathbf{C}) < 0$, $\rho\left(\boldsymbol{A}^{(\infty)}\right) < 1$, $E(|\eta_0|^s) < \infty$ for some $s > 0$ and η_0 admits a density f around 0 such that $f(y^{-1}) = o(|y|^{\delta})$ for $\delta < 1$ when $|y| \to \infty$.
Then $E|\epsilon_0|^{2s_1} < \infty$ for some $0 < s_1 \leq s$.

Sufficient conditions for the existence of $E|\epsilon_0|^{2s} < \infty$ are also available.
See Bauwens , Galli and Giot (2008) for an explicit expression of moments in the symmetric case.

Definition of the QMLE

Log-GARCH(p,q) with unknown parameter
$\boldsymbol{\theta}_0 = (\omega_0, \boldsymbol{\alpha}_{0+}, \boldsymbol{\alpha}_{0-}, \boldsymbol{\beta}_0) \in \Theta$ compact subset of $\mathbb{R}^d$, $d = 2q + p + 1$.

A QMLE is any measurable solution of

$$\hat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta}\in\Theta} \frac{1}{n} \sum_{t=r_0+1}^{n} \left\{ \frac{\epsilon_t^2}{\tilde{\sigma}_t^2(\boldsymbol{\theta})} + \log \tilde{\sigma}_t^2(\boldsymbol{\theta}) \right\},$$

where r_0 is fixed and $\tilde{\sigma}_t^2(\theta)$ is defined recursively for $t = 1, \ldots, n$, with positive initial values $\epsilon_0^2, \ldots, \epsilon_{1-q}^2, \tilde{\sigma}_0^2(\boldsymbol{\theta}), \ldots, , \tilde{\sigma}_{1-p}^2(\boldsymbol{\theta})$.

▸ On the choice of the initial values

Let the polynomials $\mathscr{A}_{\boldsymbol{\theta}}^{+}(z)=\sum_{i=1}^{q}\alpha_{i,+}z^{i}$, $\mathscr{A}_{\boldsymbol{\theta}}^{-}(z)=\sum_{i=1}^{q}\alpha_{i,-}z^{i}$ and $\mathscr{B}_{\boldsymbol{\theta}}(z)=1-\sum_{j=1}^{p}\beta_{j}z^{j}$. Write $\mathbf{C}(\boldsymbol{\theta}_0)$ instead of $(\mathbf{C}_t)$.

Strong consistency of the QMLE

Almost surely, $\widehat{\boldsymbol{\theta}}_n\to\boldsymbol{\theta}_0$ as $n\to\infty$ under the assumptions

- $\boldsymbol{\theta}_0\in\Theta$ and Θ is compact.
- $\gamma\{\mathbf{C}(\boldsymbol{\theta}_0)\}<0$ and $\forall\boldsymbol{\theta}\in\Theta$, $|\mathscr{B}_{\boldsymbol{\theta}}(z)|=0\Rightarrow|z|>1$.
- The support of η_0 contains at least two positive values and two negative values, $E\eta_0^2=1$ and $E|\log\eta_0^2|^{s_0}<\infty$ for some $s_0>0$.
- If $p>0$, $\mathscr{A}_{\boldsymbol{\theta}_0}^{+}(z)$ and $\mathscr{A}_{\boldsymbol{\theta}_0}^{-}(z)$ have no common root with $\mathscr{B}_{\boldsymbol{\theta}_0}(z)$. Moreover $\mathscr{A}_{\boldsymbol{\theta}_0}^{+}(1)+\mathscr{A}_{\boldsymbol{\theta}_0}^{-}(1)\neq 0$ and $|\alpha_{0q+}|+|\alpha_{0q+}|+|\beta_{0p}|\neq 0$.
- $E\left|\log\epsilon_t^2\right|<\infty$.

▸ Remark on the moment assumption

Asymptotic distribution

In addition to the assumptions of the consistency, assume

- $\boldsymbol{\theta}_0 \in \overset{\circ}{\Theta}$ and $\kappa_4 := E(\eta_0^4) < \infty$,
- there exists some $s_0 > 0$ such that $E\exp(s_0|\log\eta_0^2|) < \infty$, and $\rho(\boldsymbol{A}^{(\infty)}) < 1$.

▸ Interpretation of the Cramer's moment condition

Asymptotic normality of the QMLE

Under the previous assumptions, we have
$\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, (\kappa_4 - 1)\mathbf{J}^{-1})$ as $n \to \infty$, where
$\mathbf{J} = E[\nabla \log\sigma_t^2(\boldsymbol{\theta}_0)\nabla\log\sigma_t^2(\boldsymbol{\theta}_0)']$.

Testing for Log-GARCH

Let the general volatility "model"

$$\log \sigma_t^2 = \omega_0 + \sum_{i=1}^{q} \left(\alpha_{0,i+} 1_{\{\epsilon_{t-i}>0\}} + \alpha_{0,i-} 1_{\{\epsilon_{t-i}<0\}}\right) \log \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{0j} \log \sigma_{t-j}^2 + \sum_{k=1}^{\ell} \gamma_{0,k+} \eta_{t-k}^{+} + \gamma_{0,k-} \eta_{t-k}^{-},$$

of parameter $\boldsymbol{\vartheta}_0 = (\boldsymbol{\theta}_0', \boldsymbol{\gamma}_0')'$ where $\boldsymbol{\gamma}_0 = (\gamma_{01,+}, \gamma_{01,-}, \dots, \gamma_{0\ell,-})'$.

Log-GARCH against a model containing the EGARCH

$$H_0^{\gamma} : \boldsymbol{\gamma}_0 = 0 \quad \text{against} \quad H_1^{\gamma} : \boldsymbol{\gamma}_0 \neq 0.$$

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Log-GARCH against a model containing the EGARCH

$$H_0^{\gamma} : \boldsymbol{\gamma}_0 = 0 \quad \text{against} \quad H_1^{\gamma} : \boldsymbol{\gamma}_0 \neq 0.$$

For the general model, the properties of the estimator of $\boldsymbol{\vartheta}_0$ are unknown: ⇒ Wald and LR tests are not available.

Lagrange Multiplier (or score) test

Let

$$\widehat{\boldsymbol{\vartheta}}_n^c = (\widehat{\boldsymbol{\theta}}_n', \mathbf{0}_{1\times 2\ell})'$$

be the constrained (by $H_0^{\gamma} : \boldsymbol{\gamma}_0 = 0$) estimator of $\boldsymbol{\vartheta}_0$ in

$$\begin{aligned}\log \sigma_t^2(\boldsymbol{\vartheta}) &= \omega_0 + \textstyle\sum_{i=1}^q \left(\alpha_{0,i+} 1_{\{\epsilon_{t-i}>0\}} + \alpha_{0,i-} 1_{\{\epsilon_{t-i}<0\}}\right) \log \epsilon_{t-i}^2 \\ &\quad + \textstyle\sum_{j=1}^p \beta_{0j} \log \sigma_{t-j}^2(\boldsymbol{\vartheta}) + \textstyle\sum_{k=1}^{\ell} \gamma_{0,k+} \eta_{t-k}^+ + \gamma_{0,k-} \eta_{t-k}^-.\end{aligned}$$

The score has the form

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial}{\partial \boldsymbol{\vartheta}} \widetilde{\ell}_t(\widehat{\boldsymbol{\vartheta}}_n^c) = \begin{pmatrix} \mathbf{0}_{d\times 1} \\ \mathbf{S}_n \end{pmatrix}, \quad \widetilde{\ell}_t(\boldsymbol{\vartheta}) = \frac{\epsilon_t^2}{\widetilde{\sigma}_t^2(\boldsymbol{\vartheta})} + \log \widetilde{\sigma}_t^2(\boldsymbol{\vartheta}).$$

LM test

The score satisfies

$$\mathbf{S}_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (1 - \widehat{\eta}_t^2) \widehat{\mathbf{v}}_t \xrightarrow{d} \mathcal{N}(0, (\kappa_4 - 1)\mathcal{I}).$$

LM test statistic under the null

Under H_0^{γ} and the assumptions ensuring the CAN of the QMLE, and if

- the support of η_0 contains at least three positive values and three negative values,

we have

$$\mathbf{LM}_n^{\gamma} = (\widehat{\kappa}_4 - 1)^{-1} \mathbf{S}_n' \widehat{\mathcal{I}}^{-1} \mathbf{S}_n \xrightarrow{d} \chi^2_{2\ell}.$$

Testing for EGARCH(1,1)

Consider the general volatility model

$$\begin{aligned}\log\sigma_t^2 &= \omega_0+\gamma_0\eta_{t-1}+\delta_0|\eta_{t-1}|+\beta_0\log\sigma_{t-1}^2\\ &+\sum_{i=1}^q\left(\alpha_{0,i+}1_{\{\epsilon_{t-i}>0\}}+\alpha_{0,i-}1_{\{\epsilon_{t-i}<0\}}\right)\log\epsilon_{t-i}^2,\end{aligned}$$

of parameter $\boldsymbol{\vartheta}_0=(\boldsymbol{\zeta}_0',\boldsymbol{\alpha}_0')'$ where $\boldsymbol{\alpha}_0=(\boldsymbol{\alpha}_{0+}',\boldsymbol{\alpha}_{0-}')'$.
Let

$$\widehat{\boldsymbol{\vartheta}}_n^c=(\widehat{\boldsymbol{\zeta}}_n',\mathbf{0}_{1\times 2q})'$$

be the constrained (by $H_0^{\boldsymbol{\alpha}}:\boldsymbol{\alpha}_0=0$) and the score

$$\frac{1}{\sqrt{n}}\sum_{t=1}^n\frac{\partial}{\partial\boldsymbol{\vartheta}}\widetilde{\ell}_t(\widehat{\boldsymbol{\vartheta}}_n^c)=\begin{pmatrix}\mathbf{0}_{4\times 1}\\ \mathbf{T}_n\end{pmatrix}.$$

CAN of the EGARCH(1,1) QMLE

Invertibility of

$$\log \sigma_t^2(\zeta_0) = \omega_0 + \gamma_0 \eta_{t-1} + \delta_0 |\eta_{t-1}| + \beta_0 \log \sigma_{t-1}^2(\zeta_0),$$

if $\zeta_0 := (\omega_0, \gamma_0, \delta_0, \beta_0)' \in \Xi \subset \mathbb{R} \times \{\delta \geq |\gamma|\} \times \mathbb{R}^+$ and $\forall \zeta \in \Xi$,

$$E\left[\log\left(\max\left[\beta, \frac{1}{2}(\gamma\epsilon_0 + \delta|\epsilon_0|)\exp\left\{-\frac{\alpha}{2(1-\beta)}\right\} - \beta\right]\right)\right] < 0.$$

Wintenberger and Cai (2011)

The QMLE $\widehat{\zeta}_n$ over any compact set Ξ satisfying the previous invertibility condition is strongly consistent, and

$$\sqrt{n}(\widehat{\zeta}_n - \zeta_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, (\kappa_4 - 1)\mathbf{V}^{-1})$$

if $\zeta_0 \in \overset{\circ}{\Xi}$, $E(\eta_0^4) < \infty$ and $E[\{\beta_0 - \frac{1}{2}(\gamma_0\eta_0 + \delta_0 |\eta_0|)\}^2] < 1$.

LM test for EGARCH(1,1)

Recall the score

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial}{\partial \boldsymbol{\vartheta}}\widetilde{\ell}_t(\widehat{\boldsymbol{\vartheta}}_n^c)=\begin{pmatrix}\mathbf{0}_{4\times 1}\\ \mathbf{T}_n\end{pmatrix}.$$

LM test statistic under the null $H_0^{\boldsymbol{\alpha}}:\boldsymbol{\alpha}_0=0$

Under $H_0^{\boldsymbol{\alpha}}$, the assumptions of Wintenberger and Cai, and the previous assumption on the support of η_0, we have

$$\mathbf{LM}_n^{\boldsymbol{\alpha}}=(\widehat{\kappa}_4-1)^{-1}\mathbf{T}_n'\widehat{\mathscr{L}}^{-1}\mathbf{T}_n\xrightarrow{d}\chi^2_{2q}.$$

Li and Mak portmanteau statistic

Note that $\widehat{\eta}_t = \epsilon_t / \widehat{\sigma}_t$ has always zero autocorrelations, even when the volatility is misspecified: $\Rightarrow$ portmanteau tests based on residual autocorrelations are irrelevant.
The autocovariances of the squared residuals at lag h is

$$\widehat{r}_h = \frac{1}{n} \sum_{t=|h|+1}^{n} (\widehat{\eta}_t^2 - 1)(\widehat{\eta}_{t-|h|}^2 - 1), \qquad \widehat{\eta}_t^2 = \frac{\epsilon_t^2}{\widehat{\sigma}_t^2}$$

where $\widehat{\sigma}_t = \widetilde{\sigma}_t(\widehat{\boldsymbol{\theta}}_n)$. For any fixed integer m, $1 \le m < n$, consider the statistic

$$\widehat{\boldsymbol{r}}_m = \begin{pmatrix} \widehat{r}_1 \\ \vdots \\ \widehat{r}_m \end{pmatrix}.$$

Portmanteau test for adequacy of a log-GARCH

Define the $m \times d$ matrix $\widehat{\boldsymbol{K}}_m$ with row h

$$\widehat{\boldsymbol{K}}_m(h,\cdot) = \frac{1}{n} \sum_{t=h+1}^{n} (\widehat{\eta}_{t-h}^2 - 1) \nabla \log \widetilde{\sigma}_t^2(\widehat{\boldsymbol{\theta}}_n).$$

Assume the log-GARCH model with the assumptions for CAN and

- the support of η_0 contains at least three positive values or three negative values.

Portmanteau adequacy test

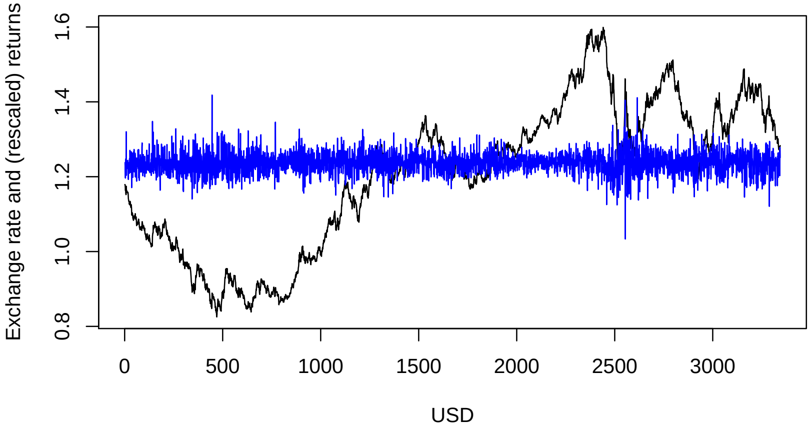
Under the previous assumptions

$$n \widehat{\boldsymbol{r}}_m' \widehat{\boldsymbol{D}}^{-1} \widehat{\boldsymbol{r}}_m \xrightarrow{\mathscr{L}} \chi_m^2,$$

with $\widehat{\boldsymbol{D}} = (\widehat{\kappa}_4 - 1)^2 \boldsymbol{I}_m - (\widehat{\kappa}_4 - 1) \widehat{\boldsymbol{K}}_m \widehat{\boldsymbol{J}}^{-1} \widehat{\boldsymbol{K}}_m'$.

Exchange and returns USD/EURO

January 5, 1999 to January 18, 2012 (3344 observations)



QMLE of the two models for exchange rate returns

Log-GARCH(1,1)

Currency	$\widehat{\omega}$	$\widehat{\alpha}_+$	$\widehat{\alpha}_-$	$\widehat{\beta}$
USD	0.024 (0.005)	0.027 (0.004)	0.016 (0.004)	0.971 (0.005)
JPY	0.051 (0.007)	0.037 (0.006)	0.042 (0.006)	0.952 (0.006)
GBP	0.032 (0.006)	0.030 (0.005)	0.029 (0.005)	0.964 (0.006)
CHF	0.057 (0.012)	0.046 (0.008)	0.036 (0.007)	0.954 (0.008)
CAD	0.021 (0.005)	0.025 (0.004)	0.017 (0.004)	0.969 (0.006)

EGARCH(1,1)

	$\widehat{\omega}$	$\widehat{\gamma}$	$\widehat{\delta}$	$\widehat{\beta}$
USD	-0.202 (0.030)	-0.015 (0.014)	0.218 (0.031)	0.961 (0.010)
JPY	-0.152 (0.021)	-0.061 (0.014)	0.171 (0.024)	0.970 (0.006)
GBP	-0.447 (0.048)	-0.029 (0.021)	0.420 (0.041)	0.913 (0.017)
CHF	-0.246 (0.046)	-0.071 (0.022)	0.195 (0.035)	0.962 (0.009)
CAD	-0.091 (0.017)	-0.008 (0.010)	0.103 (0.019)	0.986 (0.005)

p-values of portmanteau adequacy tests

Log-GARCH(1,1)

Currency	m							
	1	2	3	4	6	8	10	12
USD	0.079	0.214	0.379	0.097	0.022	0.052	0.068	0.113
~~JPY~~	0.009	0.000	0.000	0.000	0.000	0.000	0.000	0.000
GBP	0.021	0.016	0.014	0.013	0.034	0.066	0.114	0.149
~~CHF~~	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
~~CAD~~	0.003	0.013	0.013	0.004	0.009	0.028	0.025	0.038

EGARCH(1,1)

Currency	1	2	3	4	6	8	10	12
~~USD~~	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000
JPY	0.944	0.577	0.774	0.852	0.649	0.723	0.447	0.565
~~GBP~~	0.013	0.032	0.050	0.019	0.007	0.001	0.000	0.000
CHF	0.779	0.468	0.677	0.726	0.617	0.759	0.856	0.054
CAD	0.973	0.125	0.135	0.209	0.118	0.014	0.031	0.059

LM adequacy tests

EGARCH(1,1) adequacy tests under the null

portmanteau test

	m							
Iter	1	2	3	4	6	8	10	12
1	0.623	0.506	0.116	0.131	0.086	0.169	0.089	0.140
2	0.489	0.734	0.738	0.416	0.207	0.305	0.198	0.182
3	0.269	0.518	0.316	0.472	0.665	0.826	0.891	0.829
4	0.490	0.718	0.348	0.182	0.104	0.143	0.249	0.344
5	0.956	0.688	0.834	0.868	0.912	0.968	0.847	0.926

Lagrange-Multiplier test

	q							
Iter	1	2	3	4	6	8	10	12
1	0.474	0.572	0.748	0.846	0.766	0.365	0.469	0.436
2	0.992	0.833	0.900	0.690	0.793	0.624	0.698	0.429
3	0.997	0.387	0.538	0.588	0.566	0.476	0.418	0.559
4	0.106	0.254	0.193	0.327	0.284	0.534	0.207	0.394
5	0.932	0.994	0.613	0.136	0.365	0.433	0.320	0.336

Log-GARCH(1,1) adequacy tests under the null

Power of EGARCH(1,1) tests

portmanteau test

	m							
Iter	1	2	3	4	6	8	10	12
1	0.408	0.006	0.004	0.002	0.008	0.023	0.002	0.002
2	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000
3	0.369	0.231	0.023	0.019	0.001	0.000	0.000	0.000
4	0.025	0.002	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Lagrange-Multiplier test

	q							
Iter	1	2	3	4	6	8	10	12
1	0.550	0.092	0.162	0.282	0.193	0.245	0.191	0.258
2	0.110	0.023	0.047	0.078	0.126	0.164	0.163	0.293
3	0.062	0.138	0.301	0.135	0.173	0.377	0.586	0.180
4	0.801	0.722	0.494	0.253	0.204	0.234	0.198	0.258
5	0.075	0.016	0.019	0.042	0.049	0.088	0.055	0.129

Power of Log-GARCH(1,1) adequacy tests

Conclusion

- Contrary to GARCH or EGARCH models, the log-GARCH tends to produce clusters of small values.
- Stationarity and existence of (log) moments are slightly more difficult to established for the asymmetric log-GARCH than for the EGARCH (because of asymmetric effects in the persistence coefficient β).
- The CAN of the QMLE is much easier to obtain for the log-GARCH than for the EGARCH, but is a little bit more delicate than for the standard GARCH (because the volatility is not bounded away from 0).

Conclusion (continued)

- Adding EGARCH coefficients to a log-GARCH equation leads to an intractable model, which makes unavailable the Wald and LR tests.
- LM and portmanteau tests can however be used.
- For daily returns (of stock indices or exchange rates) the log-GARCH are often rejected (but we are trying duration data ...).
- For testing log-GARCH(1,1) against EGARCH(1,1) or the reverse, the portmanteau tests are more powerful than the LM tests.

Conclusion (continued)

- Adding EGARCH coefficients to a log-GARCH equation leads to an intractable model, which makes unavailable the Wald and LR tests.
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- For testing log-GARCH(1,1) against EGARCH(1,1) or the reverse, the portmanteau tests are more powerful than the LM tests.

Thanks for your attention ☺ !

Table: Autocorrelations of transformations of the CAC returns ϵ

h	1	2	3	4	5	6
$\hat{\rho}_{\epsilon}(h)$	-0.01	-0.03	-0.05	0.05	-0.06	-0.02
$\hat{\rho}_{\lvert\epsilon\rvert}(h)$	0.18	0.24	0.25	0.23	0.25	0.23
$\hat{\rho}(\epsilon_{t-h}^{+}, \lvert\epsilon_t\rvert)$	0.03	0.07	0.07	0.08	0.08	0.12
$\hat{\rho}(-\epsilon_{t-h}^{-}, \lvert\epsilon_t\rvert)$	0.18	0.20	0.22	0.18	0.21	0.15

Return

The condition $\gamma(\mathbf{C}) < 0$ is not necessary

Assume $\epsilon_t = \sigma_t \eta_t$ with

$$\log \sigma_t^2 = \omega + \alpha \log \epsilon_{t-1}^2 + \beta \log \sigma_{t-1}^2.$$

Then

$$\gamma(\mathbf{C}) < 0 \Leftrightarrow |\alpha + \beta| < 1.$$

If $\eta_0^2 = 1$ a.s. and $\alpha + \beta \neq 1$, there exists a stationary solution defined by $\epsilon_t = \exp(c/2)\eta_t$, with $c = \omega/(1 - \alpha - \beta)$.

Return

The symmetric case

In the case $\alpha_{i+} = \alpha_{i-}$, we have an ARMA-type equation of the form

$$\left\{1 - \sum_{i=1}^{r} (\alpha_i + \beta_i) B^i \right\} \log \sigma_t^2 = c + \sum_{i=1}^{q} \alpha_i B^i v_t.$$

Stationarity of the symmetric log-GARCH

Assume that $E \log^+ |\log \eta_0^2| < \infty$. There exists a (unique) strictly stationary non degenerate and non anticipative solution to the symmetric log-GARCH(p,q) model if and only if

$$z - \sum_{i=1}^{r} (\alpha_i + \beta_i) z^i = 0 \Rightarrow |z| > 1.$$

Return

Another Markovian representation

log-GARCH(1,1) case

The log-GARCH(1,1) model $\epsilon_t = \sigma_t \eta_t$ with

$$\log \sigma_t^2 = \omega + \left(\alpha_+ 1_{\{\epsilon_{t-1}>0\}} + \alpha_- 1_{\{\epsilon_{t-1}<0\}}\right) \log \epsilon_{t-1}^2 + \beta \log \sigma_{t-1}^2$$

can be written as $\boldsymbol{z}_t = \boldsymbol{C}_t \boldsymbol{z}_{t-1} + \boldsymbol{b}_t$:

$$\begin{pmatrix} 1_{\{\epsilon_t>0\}} \log \epsilon_t^2 \\ 1_{\{\epsilon_t<0\}} \log \epsilon_t^2 \\ \log \sigma_t^2 \end{pmatrix} = \begin{pmatrix} 1_{\{\eta_t>0\}} \alpha_+ & 1_{\{\eta_t>0\}} \alpha_- & 1_{\{\eta_t>0\}} \beta \\ 1_{\{\eta_t<0\}} \alpha_+ & 1_{\{\eta_t<0\}} \alpha_- & 1_{\{\eta_t<0\}} \beta \\ \alpha_+ & \alpha_- & \beta \end{pmatrix} \begin{pmatrix} 1_{\{\epsilon_{t-1}>0\}} \log \epsilon_{t-1}^2 \\ 1_{\{\epsilon_{t-1}<0\}} \log \epsilon_{t-1}^2 \\ \log \sigma_{t-1}^2 \end{pmatrix} + \begin{pmatrix} (\omega + \log \eta_t^2) 1_{\{\eta_t>0\}} \\ (\omega + \log \eta_t^2) 1_{\{\eta_t<0\}} \\ \omega \end{pmatrix}$$

or, more simply, as $\boldsymbol{\sigma}_{t,1} = \boldsymbol{A}_t \boldsymbol{\sigma}_{t-1,1} + \boldsymbol{u}_t$:

$$\begin{aligned} \log \sigma_t^2 &= \left(\alpha_+ 1_{\{\eta_{t-1}>0\}} + \alpha_- 1_{\{\eta_{t-1}<0\}} + \beta\right) \log \sigma_t^2 \\ &\quad + \left(\omega + \alpha_+ 1_{\{\eta_{t-1}>0\}} + \alpha_- 1_{\{\eta_{t-1}<0\}}\right) \log \eta_{t-1}^2. \end{aligned}$$

General case

The symmetric case

In the case $\alpha_{i+} = \alpha_{i-}$, we have

$$\begin{aligned} \rho(\boldsymbol{A}^{(1)}) < 1 \quad &\Leftrightarrow \quad \rho(\boldsymbol{A}^{(m)}) < 1 \Leftrightarrow \rho(\boldsymbol{A}^{(\infty)}) < 1 \\ &\Leftrightarrow \quad \sum_{i=1}^{r} \left|\alpha_i + \beta_i\right| < 1. \end{aligned}$$

Existence of log-moments in the symmetric case

If $\sum_{i=1}^{r} \left|\alpha_i + \beta_i\right| < 1$, then

$$E|\log \epsilon_t^2|^m < \infty \quad \text{and} \quad E|\log \sigma_t^2|^m < \infty$$

for all m such that $E|\log \eta_0^2|^m < \infty$.

◂ General case

On the choice of the initial values

One can choose $\epsilon_0, \cdots, \epsilon_{1-q}, \widetilde{\sigma}_0(\boldsymbol{\theta}), \cdots = \widetilde{\sigma}_{1-p}(\boldsymbol{\theta})$ equal to

- a constant (for example $\sqrt{2}$ for daily returns en percentage);
- a function of the parameter (for example $\exp(\omega/2)$);
- a function of the observations (for example $\sqrt{n^{-1}\sum_{t=1}^{n}\epsilon_t^2}$);
- a proxi of σ_1 (for example $\sqrt{\sum_{t=1}^{5}\epsilon_t^2/5}$);
- ...,

provided there exists a real random variable K independent of n such that

$$\sup_{\boldsymbol{\theta}\in\Theta}\left|\log\sigma_t^2(\boldsymbol{\theta}) - \log\widetilde{\sigma}_t^2(\boldsymbol{\theta})\right| < K, \quad \text{a.s. for } t = q-p+1,\dots,q.$$

Return

Remark on the moment assumption

In the symmetric log-GARCH(1,1) case, we have

$$\sigma_t^2(\boldsymbol{\theta}) = e^{\beta^{t-1}\log\sigma_1^2(\boldsymbol{\theta})}\prod_{i=0}^{t-2} e^{\beta^i\{\omega+\alpha\log\epsilon_{t-1-i}^2\}}.$$

Thus we have

$$\begin{aligned}\frac{1}{t}\log\left|\frac{1}{\sigma_t^2(\boldsymbol{\theta})}-\frac{1}{\tilde{\sigma}_t^2(\boldsymbol{\theta})}\right| &= \frac{-1}{t}\sum_{i=0}^{t-2}\beta^i\left\{\omega+\alpha\log\epsilon_{t-1-i}^2\right\} \\ &\quad +\frac{1}{t}\log\left|e^{-\beta^{t-1}\log\sigma_1^2(\boldsymbol{\theta})}-e^{-\beta^{t-1}\log\tilde{\sigma}_1^2(\boldsymbol{\theta})}\right|.\end{aligned}$$

The first term of the right-hand side of the equality tends almost surely to zero because it is bounded by $|X_t|/t$ with $E|X_t|<\infty$.

◂ Return

Interpretation of the Cramer's moment condition

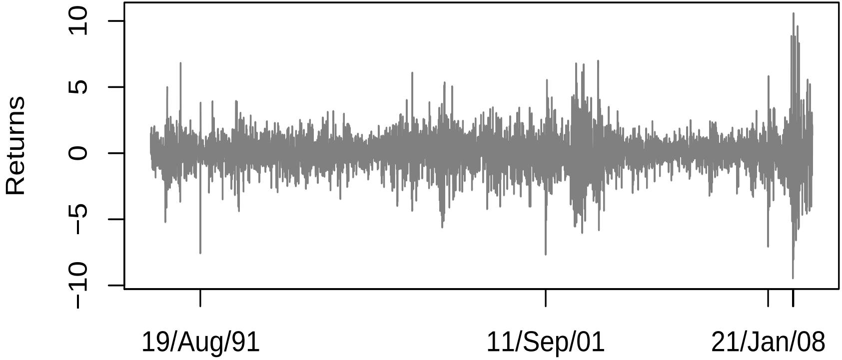
If $E(|\eta_1|^s) < \infty$ for some $s > 0$ and η_t admits a density around 0 such that $f(y^{-1}) = o(|y|^\delta)$ for $\delta < 1$ when $|y| \to \infty$ then

Cramer's moment condition

$E\exp(s_0|\log\eta_0^2|) < \infty$ for some $s_0 > 0$.

Return

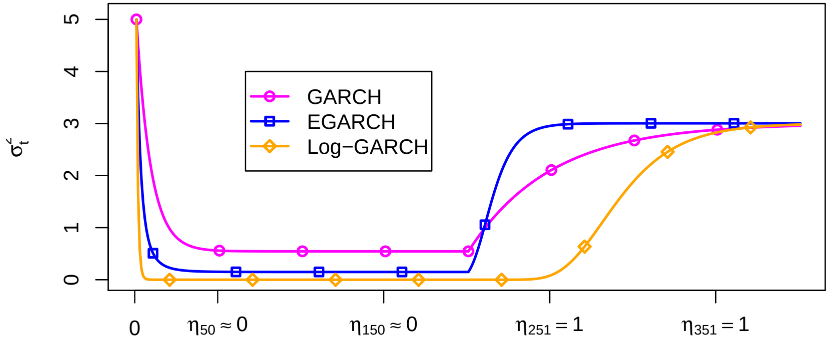
... resembles real financial series



CAC returns

Return

Effect of a sequence of small values



GARCH: $\sigma_t^2 = 0.06 + 0.09\epsilon_{t-1}^2 + 0.89\sigma_{t-1}^2$
log-GARCH: $\log \sigma_t^2 = 0.033 + 0.03 \log \epsilon_{t-1}^2 + 0.93 \log \sigma_{t-1}^2$
EGARCH: $\log \sigma_t^2 = 0.044 + 0.3|\eta_{t-1}| + 0.9 \log \sigma_{t-1}^2$

Return

Example: the log-GARCH(1,1) case

$\log \sigma_t^2 = \omega + \left(\alpha_+ 1_{\{\eta_{t-1}>0\}} + \alpha_- 1_{\{\eta_{t-1}<0\}} + \beta\right) \log \sigma_{t-1}^2 + \left(\alpha_+ 1_{\{\eta_{t-1}>0\}} + \alpha_- 1_{\{\eta_{t-1}<0\}}\right) \log \eta_{t-1}^2$

The top Lyapunov exponent is

$$\gamma(\mathbf{C}) = E \log \left| \alpha_+ 1_{\{\eta_0>0\}} + \alpha_- 1_{\{\eta_0<0\}} + \beta \right| = \log |\beta + \alpha_+|^a |\beta + \alpha_-|^{1-a},$$

where $a = P(\eta_0 > 0)$.

Stationarity of the log-GARCH(1,1)

Assume that $E \log^+ |\log \eta_0^2| < \infty$. A sufficient condition for the existence of a (unique) strictly stationary (and non anticipative) solution to the log-GARCH(1,1) model is

$$|\beta + \alpha_+|^a |\beta + \alpha_-|^{1-a} < 1.$$

Return
Symmetric case

p-values of LM adequacy tests

Currency	ℓ or q							
	1	2	3	4	6	8	10	12
Log-GARCH(1,1)								
USD	0.110	0.136	0.216	0.362	0.543	0.452	0.213	0.128
~~JPY~~	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
GBP	0.902	0.554	0.801	0.860	0.862	0.888	0.929	0.981
~~CHF~~	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
~~CAD~~	0.001	0.003	0.004	0.011	0.001	0.000	0.000	0.000
EGARCH(1,1)								
~~USD~~	0.364	0.022	0.004	0.013	0.024	0.060	0.031	0.018
JPY	0.710	0.626	0.831	0.769	0.855	0.801	0.908	0.440
GBP	0.596	0.392	0.421	0.594	0.448	0.308	0.297	0.255
CHF	0.961	0.206	0.018	0.023	0.073	0.063	0.146	0.071
CAD	0.369	0.504	0.719	0.872	0.956	0.972	0.995	0.975

Return

Log-GARCH(1,1) adequacy tests under the null

portmanteau test

	m							
Iter	1	2	3	4	6	8	10	12
1	0.722	0.068	0.088	0.119	0.251	0.258	0.102	0.128
2	0.628	0.599	0.690	0.674	0.787	0.894	0.955	0.928
3	0.338	0.590	0.788	0.764	0.671	0.816	0.773	0.710
4	0.491	0.623	0.236	0.291	0.527	0.338	0.454	0.327
5	0.057	0.133	0.257	0.370	0.374	0.594	0.757	0.631

Lagrange-Multiplier test

	q							
Iter	1	2	3	4	6	8	10	12
1	0.148	0.154	0.084	0.151	0.035	0.018	0.033	0.057
2	0.842	0.927	0.472	0.615	0.729	0.833	0.829	0.704
3	0.651	0.569	0.706	0.702	0.466	0.602	0.725	0.808
4	0.358	0.607	0.673	0.805	0.416	0.614	0.666	0.797
5	0.802	0.529	0.608	0.646	0.843	0.723	0.429	0.469

Return

Power of Log-GARCH(1,1) tests

portmanteau test

	m							
Iter	1	2	3	4	6	8	10	12
1	0.019	0.009	0.009	0.001	0.000	0.000	0.000	0.000
2	0.199	0.060	0.001	0.003	0.000	0.000	0.000	0.001
3	0.066	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.006	0.003	0.000	0.000	0.000	0.000	0.000	0.000

Lagrange-Multiplier test

	q							
Iter	1	2	3	4	6	8	10	12
1	0.290	0.522	0.742	0.154	0.422	0.368	0.181	0.322
2	0.845	0.917	0.330	0.196	0.050	0.076	0.021	0.025
3	0.315	0.544	0.339	0.495	0.516	0.479	0.397	0.463
4	0.150	0.144	0.208	0.350	0.606	0.264	0.074	0.154
5	0.933	0.915	0.983	0.979	0.622	0.542	0.649	0.603

Return