Testing strict stationarity in GARCH models

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(joint work with Jean-Michel Zakoïan)

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Motivations

- Testing for strict stationarity of financial series:
  - Standard working hypotheses: the prices p_t are nonstationary and the returns $\epsilon_t = \log p_t / p_{t-1}$ are stationary.
  - Unit root tests are available for testing nonstationarity of (p_t), but no tool for testing strict stationarity of (ϵ_t).

  $\hookrightarrow$ **Testing the stationarity of the price volatility in order to interpret the asymptotic effects of the economic shocks.**

- The statistical inference of GARCH mainly rests on the strict stationarity assumption.

  $\hookrightarrow$ **Checking if the usual inference tools are reliable.**

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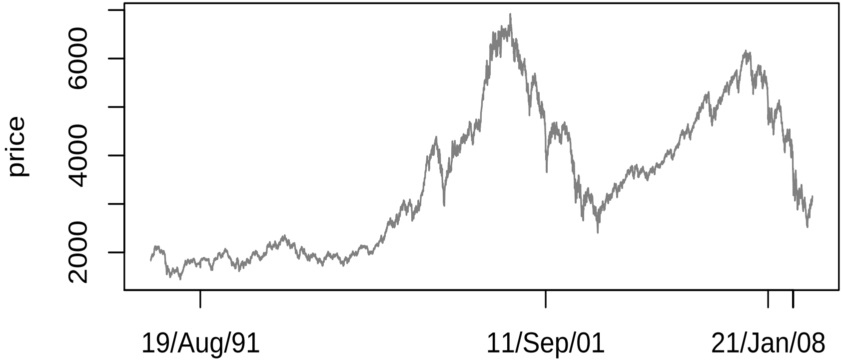
Other motivations

Stylized Facts (Mandelbrot (1963))

Non stationarity of the prices

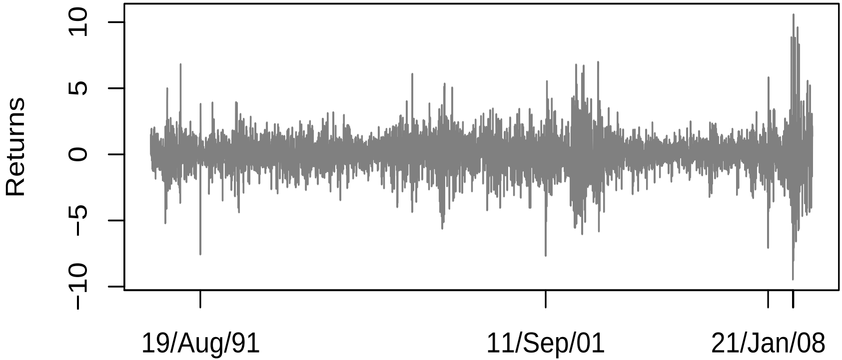


CAC 40, from March 1, 1992 to April 30, 2009

SP 500

Stylized Facts

Possible stationarity, unpredictability and volatility clustering of the returns

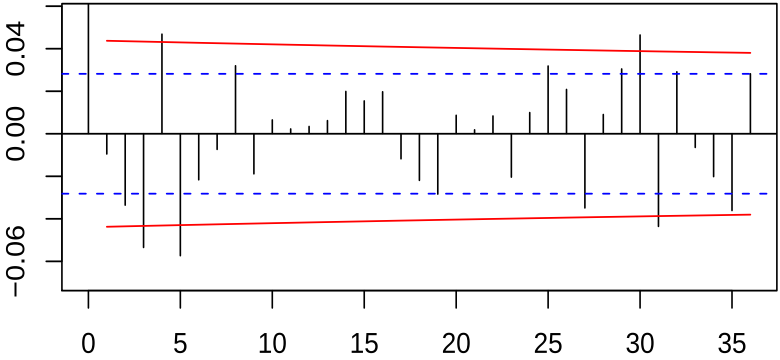


CAC 40 returns, from March 2, 1990 to February 20, 2009

SP 500 | zoom CAC40 | zoom SP500

Stylized Facts

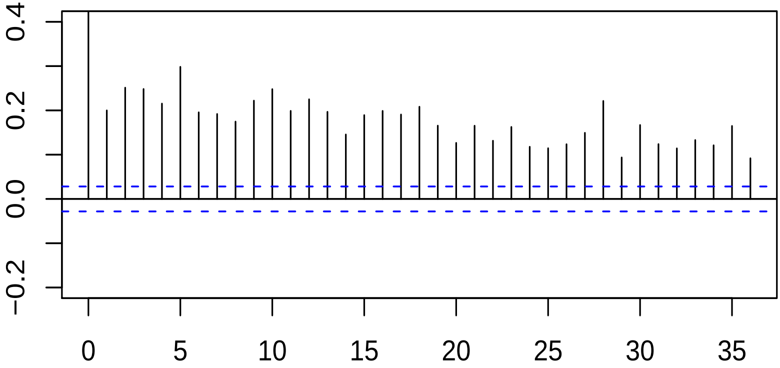
Dependence without correlation (warning: interpretation of the dotted lines)



Empirical autocorrelations of the CAC returns

SP 500
Other indices

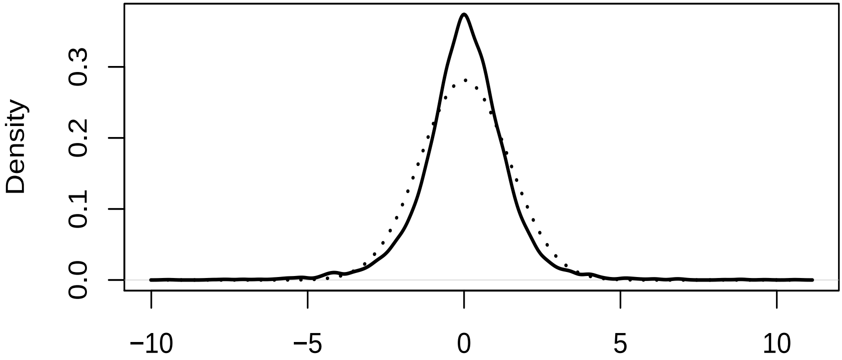
Stylized Facts
Correlation of the squares



Autocorrelations of the squares of the CAC returns

SP 500

Stylized Facts
Tail heaviness of the distributions



Density estimator for the CAC returns (normal in dotted line)

SP 500

Main properties of daily stock returns

- Unpredictability of the returns (martingale difference assumption), but non-independence.
- Strong positive autocorrelations of the squares or of the absolute values (even for large lags).
- Volatility clustering.
- Leptokurticity of the marginal distribution.
- Decreases of prices have an higher impact on the future volatility than increases of the same magnitude (leverage effects).

Leverage effects

- Seasonalities.

Volatility Models

Almost all the volatility models are of the form

$$\epsilon_t = \sigma_t \eta_t$$

where (η_t) is iid (0,1), $\sigma_t > 0$, σ_t and η_t are independent.

For GARCH-type (Generalized Autoregressive Conditional Heteroskedasticity) models, $\sigma_t \in \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \ldots)$.

See Bollerslev (Glossary to ARCH (GARCH), 2009) for an impressive list of more than one hundred GARCH-type models.

Definition: GARCH(p, q)

Definition (Engle (1982), Bollerslev (1986))

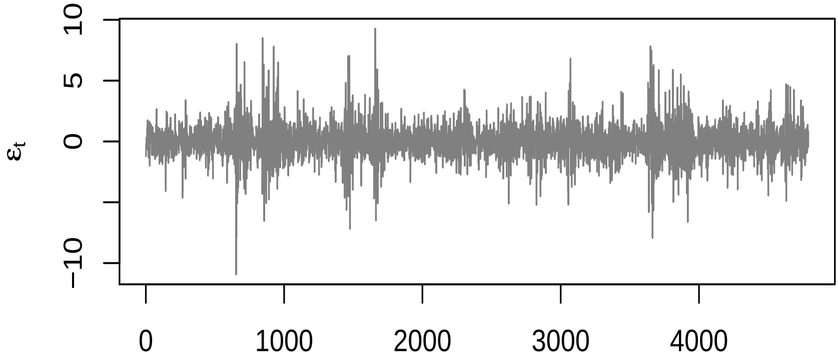
$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t^2 = \omega_0 + \sum_{i=1}^{q} \alpha_{0i} \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{0j} \sigma_{t-j}^2, \quad \forall t \in \mathbb{Z} \end{cases}$$

where

$$(\eta_t) \text{ iid}, \quad E\eta_t = 0, \quad E\eta_t^2 = 1, \quad \omega_0 > 0, \quad \alpha_{0i} \geq 0, \quad \beta_{0j} \geq 0.$$

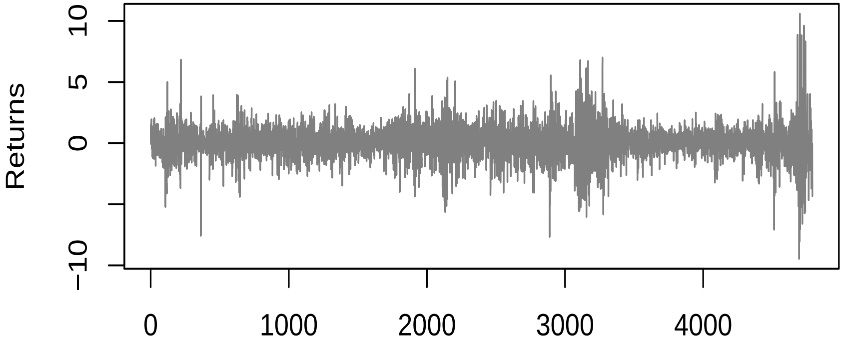
Return

GARCH(1,1) simulation



$$\epsilon_t = \sigma_t \eta_t,\ \eta_t \text{ iid } St_5,\ \sigma_t^2 = 0.033 + 0.090\epsilon_{t-1}^2 + 0.893\sigma_{t-1}^2,$$
$$t = 1, \ldots, n = 4791$$

The previous GARCH(1,1) simulation resembles real financial series



CAC returns

Few references on QML estimation for GARCH:

- **ARCH(q) or GARCH(1,1):** Weiss (Econometric Theory, 1986), Lee and Hansen (Econometric Theory, 1994), Lumsdaine (Econometrica, 1996),
- **GARCH(p, q):** Berkes, Horváth and Kokoszka (Bernoulli, 2003), Francq and Zakoïan (Bernoulli, 2004), Hall and Yao (Econometrica, 2003), Mikosch and Straumann (Ann. Statist., 2006).
- **More general stationary GARCH models:** Straumann and Mikosch (Ann. Statist., 2006), Robinson and Zaffaroni (Ann. Statist., 2006), Bardet and Wintenberger (Ann. Statist., 2009).
- **Explosive ARCH(1) and GARCH(1,1):** Jensen and Rahbek (Econometrica, 2004 and Econometric Theory, 2004).

▸ GARCH

Outline

Strict Stationarity of the GARCH(1,1) Model

GARCH(1,1) Model:

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, & t = 1, 2, \dots \\ h_t = \omega_0 + \alpha_0 \epsilon_{t-1}^2 + \beta_0 h_{t-1} \end{cases}$$

with initial values ϵ_0 and $h_0 \geq 0$, where $\omega_0 > 0$, $\alpha_0, \beta_0 \geq 0$, and (η_t) iid (0,1) with $P(\eta_1^2 = 1) < 1$.

Necessary and Sufficient Strict Stationarity Condition:

$$\gamma_0 < 0,$$

where $\gamma_0 = E \log\left(\alpha_0 \eta_1^2 + \beta_0\right)$.

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Probabilistic framework (Nelson (1990), Klüppelberg, Lindner and Maller (2004))

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, & t = 1, 2, \ldots \\ h_t = \omega_0 + a_0(\eta_t)h_{t-1}, & \text{with } a_0(x) = \alpha_0 x^2 + \beta_0 \text{ and initial values.} \end{cases}$$

- $\gamma_0 < 0$: the effect of the initial values vanishes asymptotically:

  $$h_t - \sigma_t^2 \to 0 \text{ almost surely as } t \to \infty,$$

  where σ_t^2 is a stationary process involving the infinite past.
- $\gamma_0 > 0$: $h_t \to \infty$, almost surely as $t \to \infty$.
- $\gamma_0 = 0$: $h_t \to \infty$, in probability as $t \to \infty$.

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Stationarity and explosiveness

Nonstationarity in GARCH $\Leftrightarrow$ explosiveness

$$h_t \to \infty \Rightarrow \epsilon_t^2 \to \infty \text{ when } E|\log \eta_t^2| < \infty$$

Interpretation in terms of persistence of shocks

Almost surely, for any i

- $$\lim_{t\to\infty} \frac{\partial h_t}{\partial \eta_i} = 0$$
  when $\gamma_0 < 0$ (temporary effect),
- $$\lim_{t\to\infty} \frac{\partial h_t}{\partial \eta_i} = \text{sign}(\eta_i) \times \infty$$
  when $\gamma_0 > 0$ (explosive effect),
- $$\limsup_{t\to\infty} \frac{\partial h_t}{\partial |\eta_i|} = +\infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{\partial h_t}{\partial |\eta_i|} = 0$$
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Definition of the standard (unrestricted) QMLE

$\theta = (\omega, \alpha, \beta)' \in \Theta$ compact subset of $(0, \infty)^3$.

A QMLE is any measurable solution of

$$\hat{\theta}_n = (\hat{\omega}_n, \hat{\alpha}_n, \hat{\beta}_n)' = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \left\{ \frac{\epsilon_t^2}{\sigma_t^2(\theta)} + \log \sigma_t^2(\theta) \right\},$$

where $\sigma_t^2(\theta) = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2(\theta)$ for $t = 1, \ldots, n$ (+ init. val.).

Remark: This is not the constrained estimator

studied by Jensen and Rahbek (2004, 2006):

$$(\hat{\alpha}_n^c(\omega), \hat{\beta}_n^c(\omega))' = \arg \min_{(\alpha,\beta) \in \Theta_{\alpha,\beta}} \frac{1}{n} \sum_{t=1}^{n} \left\{ \frac{\epsilon_t^2}{\sigma_t^2(\theta)} + \log \sigma_t^2(\theta) \right\}$$

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Consistency of the QMLE of (α_0, β_0)

- **Stationary case:** if $\gamma_0 < 0$ and $\beta < 1$ for all $\theta \in \Theta$,

$$\hat{\theta}_n \to \theta_0 = (\omega_0, \alpha_0, \beta_0)' \qquad \text{a.s. as } n \to \infty.$$

- **Nonstationary case I:** if $\gamma_0 > 0$ and $P(\eta_1 = 0) = 0$,

$$(\hat{\alpha}_n, \hat{\beta}_n) \to (\alpha_0, \beta_0) \qquad \text{a.s. as } n \to \infty,$$

▸ Idea of the proof

- **Nonstationary case II:** if $\gamma_0 = 0$, $P(\eta_1 = 0) = 0$ and there exists $p > 1$ such that $\beta < \| 1/a_0(\eta_1) \|_p^{-1}$ for all $\theta \in \Theta$,

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Contrary to the QMLE, the constrained QMLE of (α_0, β_0) is not universally consistent

When $\gamma_0 < 0$ and $E\epsilon_t^4 < \infty$, if $\omega \neq \omega_0$ the constrained QMLE

$(\hat{\alpha}_n^c(\omega), \hat{\beta}_n^c(\omega))$ **does not converge** in probability to (α_0, β_0).

Inconsistency of the QMLE of ω_0 in the case $\gamma_0 > 0$

Assume $\eta_t \sim \mathcal{N}(0,1)$ and Θ contains two arbitrarily close points $\theta = (\omega, \alpha, \beta)$ and $\theta^* = (\omega^*, \alpha, \beta)$ such that $E \log(\alpha\eta_t^2 + \beta) > 0$ and $\omega \neq \omega^*$.
Then there exists **no consistent estimator** of $\theta_0 \in \Theta$.

Asymptotic normality of the QMLE

- Stationary case: if $\gamma_0 < 0$, $\kappa_\eta = E\eta_1^4 \in (1, \infty)$, θ_0 belongs to the interior $\overset{\circ}{\Theta}$ of Θ and $\beta < 1$ for all $\theta \in \Theta$,

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{d} \mathcal{N}\left\{0, (\kappa_\eta - 1)J^{-1}\right\}, \quad \text{as } n \to \infty.$$

- Nonstationary cases I and II (under a technical assumption): if $\gamma_0 \geq 0$, $\kappa_\eta \in (1, \infty)$ $E|\log \eta_1^2| < \infty$ and $\theta_0 \in \overset{\circ}{\Theta}$,

$$\sqrt{n}\left(\hat{\alpha}_n - \alpha_0, \hat{\beta}_n - \beta_0\right) \xrightarrow{d} \mathcal{N}\left\{0, (\kappa_\eta - 1)I^{-1}\right\}, \quad \text{as } n \to \infty.$$

Forms of I and J

Technical Assumption required in the case $\gamma_0 = 0$:

When t tends to infinity,

$$E\left(\frac{1}{1+Z_1+Z_1Z_2+\cdots+Z_1\ldots Z_{t-1}}\right)=o\left(\frac{1}{\sqrt{t}}\right)$$

where $Z_t=\alpha_0\eta_t^2+\beta_0$.

Remark: $\gamma_0 = E\log Z_t = 0$ entails $EZ_t \geq 1$, so

$$E\left(1+Z_1+Z_1Z_2+\cdots+Z_1\ldots Z_{t-1}\right)\geq t.$$

Asymptotic Variance of $(\hat{\alpha}_n, \hat{\beta}_n)$

$$\sqrt{n}\left(\hat{\alpha}_n - \alpha_0, \hat{\beta}_n - \beta_0\right)' \xrightarrow{d} \mathcal{N}\left\{0, (\kappa_\eta - 1)I_*^{-1}\right\},$$

with

$$I_* = \begin{cases} J_{\alpha\beta,\alpha\beta} - J_{\alpha\beta,\omega}J_{\omega,\omega}^{-1}J_{\omega,\alpha\beta}, & \text{when } \gamma_0 < 0 \\ I, & \text{when } \gamma_0 \geq 0. \end{cases}$$

When $\gamma_0 < 0$, a natural empirical estimator of I_* is

$$\hat{I}_* = \hat{J}_{\alpha\beta,\alpha\beta} - \hat{J}_{\alpha\beta,\omega}\hat{J}_{\omega,\omega}^{-1}\hat{J}_{\omega,\alpha\beta}.$$

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Universal Estimator of the Asymptotic Variance of $(\hat{\alpha}_n, \hat{\beta}_n)$

Let $\hat{\kappa}_\eta = n^{-1} \sum_{t=1}^{n} \hat{\eta}_t^4$ be the empirical kurtosis of η_t.

Under the previous assumptions, whatever γ_0, we have

$$\hat{\kappa}_\eta \to \kappa_\eta.$$

Moreover, as $n \to \infty$,

- if $\gamma_0 < 0$: $\hat{I}_* \to I_*$ a.s
- if $\gamma_0 > 0$: $\hat{I}_* \to I$ a.s.
- if $\gamma_0 = 0$: $\hat{I}_* \to I$ in probability.

Therefore, $(\hat{\kappa}_\eta - 1)\hat{I}_*^{-1}$ **is always a consistent estimator of the asymptotic variance of the QMLE of** (α_0, β_0).

Testing (α_0, β_0) without imposing $\gamma_0 < 0$

Consider the testing problem

$$H_0: \ a\alpha_0 + b\beta_0 \leq c \qquad \text{against} \qquad H_1: \ a\alpha_0 + b\beta_0 > c,$$

where a, b, c are given numbers.

Under the previous assumptions,

the test defined by the critical region

$$\left\{ \frac{\sqrt{n}(a\hat{\alpha}_n + b\hat{\beta}_n - c)}{\sqrt{(\hat{\kappa}_\eta - 1)(a,b)\hat{I}_*^{-1}(a,b)'}} > \Phi^{-1}(1 - \underline{\alpha}) \right\}$$

has the asymptotic significance level $\underline{\alpha}$ and is consistent.

Testing for Strict Stationarity and for Nonstationarity

Consider the testing problems

$$H_0 : \gamma_0 < 0 \qquad \text{against} \qquad H_1 : \gamma_0 \geq 0,$$

and

$$H_0 : \gamma_0 \geq 0 \qquad \text{against} \qquad H_1 : \gamma_0 < 0.$$

Under the previous assumptions, with $\sigma_u^2 = \text{var} \log(\alpha_0 \eta_1^2 + \beta_0)$ and $\hat{\gamma}_n := n^{-1} \sum_{t=1}^{n} \log \left(\hat{\alpha}_n \hat{\eta}_t^2 + \hat{\beta}_n \right)$, we have

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) \xrightarrow{d} \mathcal{N}\left(0, \sigma_\gamma^2\right)$$

$$\text{where } \sigma_\gamma^2 = \begin{cases} \sigma_u^2 + \text{positive constant} & \text{when } \gamma_0 < 0, \\ \sigma_u^2 & \text{when } \gamma_0 \geq 0. \end{cases}$$

Testing for Strict Stationarity and for Nonstationarity

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Testing the Null of Strict Stationarity

For testing

$$H_0 : \gamma_0 < 0 \qquad \text{against} \qquad H_1 : \gamma_0 \geq 0,$$

the test

$$C^{ST} = \left\{ T_n := \sqrt{n}\frac{\hat{\gamma}_n}{\hat{\sigma}_u} > \Phi^{-1}(1 - \underline{\alpha}) \right\}$$

has its asymptotic significance level bounded by $\underline{\alpha}$, has the asymptotic probability of rejection $\underline{\alpha}$ under $\gamma_0 = 0$, and is consistent for all $\gamma_0 > 0$.

Testing the Null of Nonstationarity

For testing

$$H_0 : \gamma_0 \geq 0 \qquad \text{against} \qquad H_1 : \gamma_0 < 0,$$

the test

$$C^{NS} = \left\{ T_n = \sqrt{n}\frac{\hat{\gamma}_n}{\hat{\sigma}_u} < \Phi^{-1}(\underline{\alpha}) \right\}$$

has its asymptotic significance level bounded by $\underline{\alpha}$, has the asymptotic probability of rejection $\underline{\alpha}$ under $\gamma_0 = 0$, and is consistent for all $\gamma_0 < 0$.

Regularity assumptions on η_t

Assume that η_t has a density f with third-order derivatives, that

$$\lim_{|y|\to\infty} y^2 f'(y) = 0,$$

and that for some positive constants K and δ

$$|y|\left|\frac{f'}{f}(y)\right| + y^2\left|\left(\frac{f'}{f}\right)'(y)\right| + y^2\left|\left(\frac{f'}{f}\right)''(y)\right| \leq K\left(1+|y|^\delta\right),$$

$$E|\eta_1|^{2\delta} < \infty.$$

These regularity conditions entail the existence of the Fisher information for scale

$$\iota_f = \int \{1 + yf'(y)/f(y)\}^2 f(y)dy < \infty.$$

LAN under Strict Stationarity

Drost and Klaassen (1997)

Around $\theta_0 \in \overset{\circ}{\Theta}$, let a sequence of local parameters

$$\theta_n = \theta_0 + \tau_n/\sqrt{n},$$

where (τ_n) is a bounded sequence of $\mathbb{R}^2$. Under $\gamma_0 < 0$, it is known that the log-likelihood ratio

$$\Lambda_{n,f}(\theta_n, \theta_0) = \log \frac{L_{n,f}(\theta_n)}{L_{n,f}(\theta_0)}$$

satisfies the LAN property

$$\Lambda_{n,f}(\theta_n, \theta_0) = \tau_n' S_{n,f}(\theta_0) - \frac{1}{2}\tau_n' \mathfrak{I}_f \tau_n + o_{P_{\theta_0}}(1), \quad S_{n,f}(\theta_0) \xrightarrow{d} \mathcal{N}\{0, \mathfrak{I}_f\}$$

under P_{θ_0} as $n \to \infty$.

LAN without Stationarity Constraints

When $\theta_0 \in \overset{\circ}{\Theta}$, and under the regularity assumptions on f, we have the **LAN property (regardless of the sign of γ_0).** When $\gamma_0 \geq 0$, the Fisher information is the degenerate matrix

$$\mathfrak{I}_f = \frac{\iota_f}{4} \begin{pmatrix} 0 & 0 \\ 0 & \alpha_0^{-2} \end{pmatrix}.$$

LAP of the Test of $H_0 : \alpha_0 \leq \alpha^*$

The test defined by the critical region

$$C^{\alpha^*} = \left\{ \frac{\sqrt{n}(\hat{\alpha}_n - \alpha^*)}{\sqrt{(\hat{\kappa}_\eta - 1)/\hat{I}_*}} > \Phi^{-1}(1 - \underline{\alpha}) \right\}$$

where

$$\hat{I}_* = \hat{\mu}_n(2,2) - \frac{\hat{\mu}_n^2(1,2)}{\hat{\mu}_n(0,2)}, \qquad \text{with } \hat{\mu}_n(p,q) = \frac{1}{n}\sum_{t=1}^{n} \frac{\epsilon_t^{2p}}{(\hat{\omega}_n + \hat{\alpha}_n \epsilon_t^2)^q},$$

has the asymptotic significance level $\underline{\alpha}$ and is consistent.

LAP of the Test of $H_0 : \alpha_0 \leq \alpha^*$

Denote by $P_{n,\tau}^{\alpha^*}$, where $\tau = (\tau_1, \tau_2)'$, the distribution of the observations $(\epsilon_1, \dots, \epsilon_n)$ when the parameter is of the form

$$\theta_n^{\alpha^*} = (\omega_0, \alpha^*)' + \tau/\sqrt{n}, \quad \tau_2 > 0.$$

The LAP of the C^{α^*}-test is given by

$$\lim_{n\to\infty} P_{n,\tau}^{\alpha^*}\left(\mathrm{C}^{\alpha^*}\right) = \Phi\left\{\frac{\tau_2}{\sqrt{(\kappa_\eta - 1)/I_*}} - \Phi^{-1}(1-\underline{\alpha})\right\},$$

where $I_* = 1/\alpha^{*2}$ when $E\log\alpha^*\eta_1^2 \geq 0$ and I_* is more complicated when $E\log\alpha^*\eta_1^2 < 0$.

Optimality of the Test of $H_0 : \alpha_0 \leq \alpha^*$

The optimal test of $H_0 : \alpha_0 \leq \alpha^*$ has the LAP

$$\tau_2 \to \Phi\left(\frac{\tau_2}{\sqrt{4/\iota_f I_*}} - \Phi^{-1}(1-\underline{\alpha})\right).$$

The test C^{α^*} is optimal iff

$$f(y) = \frac{a^a}{\Gamma(a)} e^{-ay^2} |y|^{2a-1}, \quad a > 0, \quad \Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

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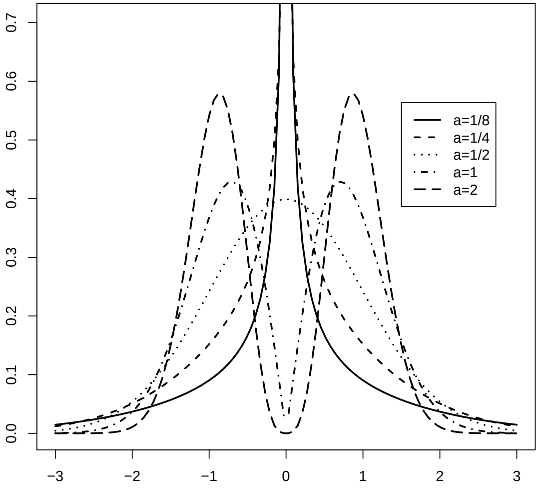
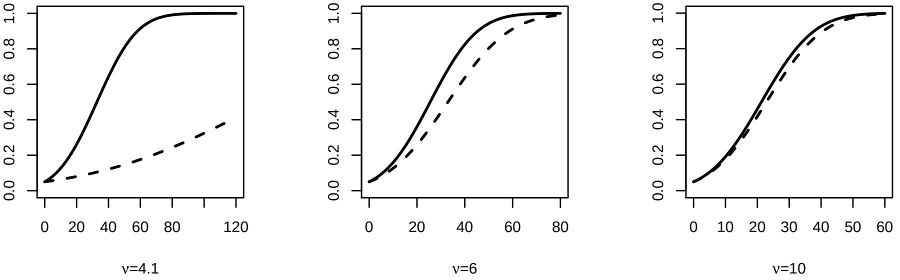
Densities of η_t for which the test C^{α^*} is asymptotically locally optimal



Optimal LAP (in full line) and LAP of the C^{α^*}-test (in dotted line) for testing $H_0 : \alpha_0 < \alpha^*$ when $\eta_t \sim St_\nu$ (standardized), with α^* such that $\gamma_0 = 0$ when $\alpha_0 = \alpha^*$.

The C^{α^*}-test is optimal in the gaussian case



Local Asymptotic Powers of the Stationarity Tests

Let $\theta_0 = (\omega_0, \alpha_0)'$ such that $\alpha_0 = \exp(-E \log \eta_t^2)$.

Let $\tau = (\tau_1, \tau_2)'$. Denote by $P_{n,\tau}$ the distribution of the observations $(\epsilon_1, \dots, \epsilon_n)$ when the parameter is

$$\left(\omega_0 + \frac{\tau_1}{\sqrt{n}},\ \alpha_0 + \frac{\tau_2}{\sqrt{n}}\right)'.$$

The LAP of the stationarity tests are given by

$$\lim_{n\to\infty} P_{n,\tau}\left(C^{ST}\right) = \Phi\left\{\frac{\tau_2}{\alpha_0\sigma_u} - \Phi^{-1}(1-\underline{\alpha})\right\}, \quad \tau_2 > 0$$

and

$$\lim_{n\to\infty} P_{n,\tau}\left(C^{NS}\right) = \Phi\left\{\Phi^{-1}(\underline{\alpha}) - \frac{\tau_2}{\alpha_0\sigma_u}\right\}, \quad \tau_2 < 0.$$

Optimal Local Asymptotic Power of the Strict Stationarity Test

The optimal ST-test of $H_0 : \gamma_0 < 0$ has the LAP

$$\tau_2 \to \Phi\left(\frac{\tau_2}{\sqrt{4\alpha_0^2/\iota_f}} - \Phi^{-1}(1-\underline{\alpha})\right).$$

The test C^{ST} (or C^{NS}) is optimal iff

$$f(y) = \frac{1}{2\sqrt{|\delta|\pi e^{-\delta/4}}} e^{\frac{(\log|y|)^2}{\delta}} y^{-2}, \quad \delta < 0.$$

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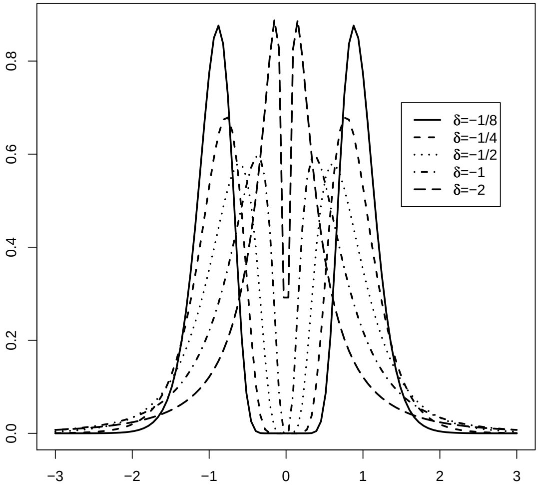
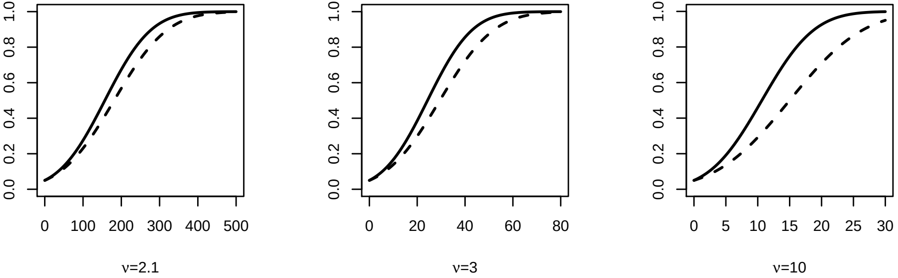
Densities of η_t for which the C^{ST} (or C^{NS}) test is asymptotically locally optimal



Optimal LAP (in full line) and LAP of the C^{ST}-test (in dotted line) when $\eta_t \sim St_\nu$ (standardized).

The C^{ST}-test is not optimal in the gaussian case



Application to Non Linear GARCH
Augmented GARCH models

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, & t = 1, 2, \dots \\ h_t = \omega(\eta_{t-1}) + a(\eta_{t-1})h_{t-1} \end{cases}$$

with $\omega : \mathbb{R} \to [\underline{\omega}, +\infty)$, for some $\underline{\omega} > 0$, and $a : \mathbb{R} \to \mathbb{R}^+$.

- Standard GARCH(1,1) when $\omega(\cdot) = \omega$ and $a(x) = \alpha_0 x^2 + \beta_0$;
- GJR model when $a(x) = \alpha_1(\max\{x, 0\})^2 + \alpha_2(\min\{x, 0\})^2 + \beta_0$.

Strict stationarity condition:

$$\Gamma := E \log a(\eta_t) < 0.$$

Application to Non Linear GARCH
Augmented GARCH models

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Application to Non Linear GARCH
Augmented GARCH models

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- GJR model when $a(x) = \alpha_1(\max\{x, 0\})^2 + \alpha_2(\min\{x, 0\})^2 + \beta_0$.

Strict stationarity condition:

$$\Gamma := E \log a(\eta_t) < 0.$$

Behavior of the test statistics when the GARCH(1,1) is misspecified

Under some regularity assumptions,

the statistics built with **the standard GARCH(1,1) model** satisfy:

If $\Gamma > 0$ then

$$\hat{\gamma}_n \to \Gamma, \quad \text{and} \quad \hat{\sigma}_u^2 \to \sigma_*^2 > 0, \quad a.s.$$

If $\Gamma < 0$ then

$$\hat{\gamma}_n \to \Gamma^* < 0, \quad \text{and} \quad \hat{\sigma}_u^2 \to \text{Var} \log \left\{ \alpha^* \frac{\epsilon_t^2}{\sigma_t^2(\theta^*)} + \beta^* \right\} > 0, \quad a.s.$$

Behavior of the Standard GARCH(1,1) Strict Stationarity Tests Applied to Augmented GARCH Processes

Under the previous assumptions,as $n \to \infty$,

if $\Gamma > 0$ then

$$P(C^{NS}) \to 0 \quad \text{and} \quad P(C^{ST}) \to 1,$$

if $\Gamma < 0$ then

$$P(C^{ST}) \to 0 \quad \text{and} \quad P(C^{NS}) \to 1,$$

if $\Gamma = 0$ then

$$P(C^{ST}) \to ? \quad \text{and} \quad P(C^{NS}) \to ?.$$

Bias and MSE for the QMLE over $1,000$ replications

$\eta_t \sim \mathcal{N}(0,1)$ and $\theta_0 = (1, 0.5, 0.6)$ (ST) or $\theta_0 = (1, 0.7, 0.6)$ (NS)

	ST ($\gamma_0 = -0.038$)			NS ($\gamma_0 = 0.078$)		
	ω	α	β	ω	α	β
$n = 200$						
Bias	-0.34	0.01	0.01	-0.51	0.02	0.02
MSE	1.10	0.02	0.02	**3.77**	0.03	0.03
$n = 4000$						
Bias	-0.03	0.00	0.00	-0.51	0.00	0.00
MSE	0.03	0.00	0.00	**4.95**	0.00	0.00

$H_0 : \beta_0 \leq 0.7$ Against $H_1 : \beta_0 > 0.7$

Nominal level 5%, $\eta_t \sim \mathrm{St}_7$ and $\alpha_0 = 0.2$ ($(\alpha_0, \beta_0) = (0.2, 0.7)$ corresponding to a stationary process)

β_0	0.61	0.64	0.67	0.70	0.73	0.76	0.79
$n = 500$	3.5	4.3	5.2	8.9	12.6	26.8	49.6
$n = 2,000$	0.3	0.6	1.8	6.8	18.3	53.1	91.5
$n = 4,000$	0.2	0.3	1.0	5.5	27.7	76.9	99.0

$H_0 : \beta_0 \leq 0.7$ Against $H_1 : \beta_0 > 0.7$

Nominal level 5%, $\eta_t \sim \mathrm{St}_7$ and $\alpha_0 = 0.5$ ($(\alpha_0, \beta_0) = (0.5, 0.7)$ corresponding to a non stationary process)

β_0	0.61	0.64	0.67	0.70	0.73	0.76	0.79
$n = 500$	0.3	0.5	2.8	9.9	25.5	47.7	67.2
$n = 2,000$	0.0	0.0	0.1	6.2	41.6	81.8	97.0
$n = 4,000$	0.0	0.0	0.1	6.1	61.0	96.2	99.7

Relative frequency of rejection for the test C^{ST}

The parameter $(\alpha_0, \beta_0) = (0.2575, 0.8)$ corresponds to $\gamma_0 = 0$

α_0	0.18	0.20	0.22	0.2575	0.28	0.30	0.31
$n = 500$	0.0	0.0	0.1	7.5	27.8	61.4	75.2
$n = 2,000$	0.0	0.0	0.0	6.3	67.8	98.6	99.9
$n = 4,000$	0.0	0.0	0.0	5.3	92.4	100.0	100.0

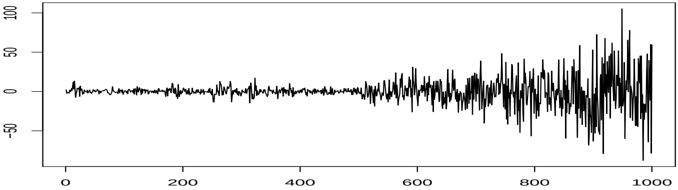
Relative frequency of rejection for the test C^{ST}

As the previous table, but the DGP is a GJR model, $\alpha_1 = 0.2575$ corresponding to $\Gamma = 0$

α_1	0.18	0.20	0.22	0.2575	0.28	0.30	0.31
$n = 500$	0.1	0.1	1.1	7.8	15.8	32.7	35.2
$n = 2,000$	0.0	0.0	0.1	6.6	31.7	65.8	77.4
$n = 4,000$	0.0	0.0	0.0	5.6	45.1	87.7	96.1

Other simulation experiments

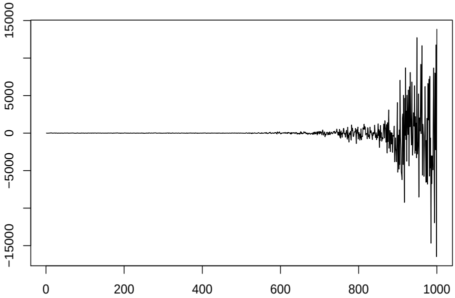
A stationary GARCH followed by another stationary GARCH



$$h_t = 1 + 0.5\epsilon_t^2 + 0.5h_{t-1} \text{ for } t = 1, \dots, 500 \text{ and}$$
$$h_t = 1 + 0.05\epsilon_t^2 + 0.95h_{t-1} \text{ for } t = 501, \dots, 1000$$

$$\hat{\alpha}_n = 0.229, \quad \hat{\beta}_n = 0.808, \quad \hat{\gamma}_n = -0.442 \text{ (p-val=0.670)}$$

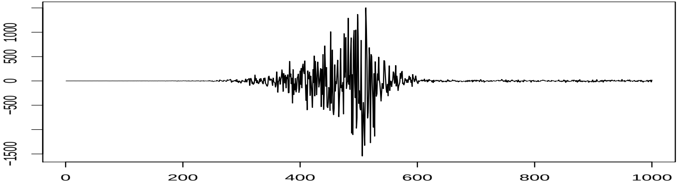
A stationary GARCH followed by an explosive GARCH



$h_t = 10 + 0.05\epsilon_t^2 + 0.89h_{t-1}$ for $t = 1, \dots, 500$ and
$h_t = 0.001 + 0.14\epsilon_t^2 + 0.91h_{t-1}$ for $t = 501, \dots, 1000$

$\hat{\alpha}_n = 0.124, \quad \hat{\beta}_n = 0.898, \quad \hat{\gamma}_n = 2.088$ (p-val=0.018)

An explosive GARCH followed by a stationary GARCH



$h_t = 0.001 + 0.14\epsilon_t^2 + 0.91h_{t-1}$ for $t = 1, \ldots, 500$ and
$h_t = 10 + 0.05\epsilon_t^2 + 0.89h_{t-1}$ for $t = 501, \ldots, 1000$

$\hat{\alpha}_n = 0.182, \quad \hat{\beta}_n = 0.850, \quad \hat{\gamma}_n = 0.981$ (p-val=0.163)

Strict stationarity test statistic T_n on daily indices (from 1990 to 2009)

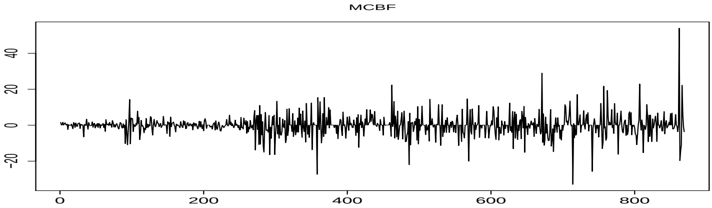
Asymptotically, $T_n \sim \mathcal{N}(0,1)$ when $\gamma_0 = 0$, tends to $-\infty$ when $\gamma_0 < 0$, and tends to $+\infty$ when $\gamma_0 > 0$

CAC	DAX	DJA	FTSE	Nasdaq	Nikkei	SMI	SP500
-14.5	-15.8	-15.1	-10.7	-8.5	-15.4	-23	-11.1

T_n and $p-$values of the non stationarity test for individual stock returns

	ICGN	MCBF	KVA	BTC	CCME
n	928	869	1222	911	469
$\hat{\alpha}_n$	0.559	0.024	0.147	0.500	0.416
$\hat{\beta}_n$	0.713	0.979	0.926	0.766	0.748
T_n	-1.597	0.100	1.209	0.052	0.123
p-val	0.055	0.540	0.887	0.521	0.549

Graph of an explosive series of returns



Log-returns (in %) of the MCBF stock series

Conclusion

- For a GARCH(1,1), the standard QMLE of (α_0, β_0) is CAN, even when $\gamma_0 \geq 0$.
- It is impossible to consistently estimate ω_0 when $\gamma_0 > 0$.
- A specific behavior is obtained at the boundary of the stationarity region: when $\gamma_0 = 0$.

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Conclusion (continued)

- It is possible to consistently estimate the asymptotic variance of $(\hat{\alpha}_n, \hat{\beta}_n)$, without knowing if $\gamma_0 < 0$ or not.
- It is therefore possible to test the value of (α_0, β_0) even in the nonstationary case.
- It is also possible to develop strict stationarity tests (shocks effect and inference validity).
- The strict stationarity tests developed for the standard GARCH(1,1) also work for more general GARCH

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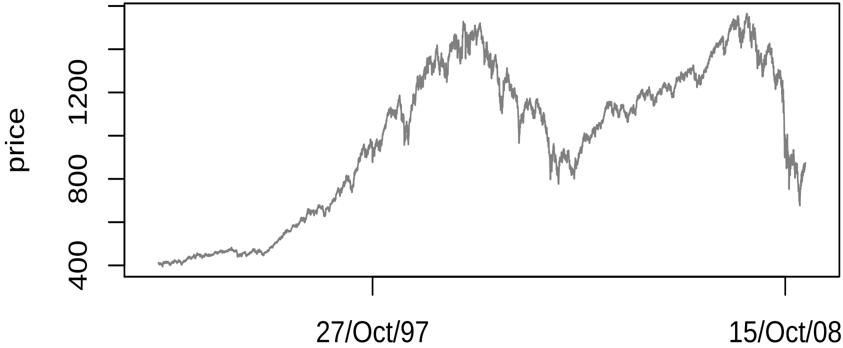
Stylized Facts (Mandelbrot (1963))

Non stationarity of the prices

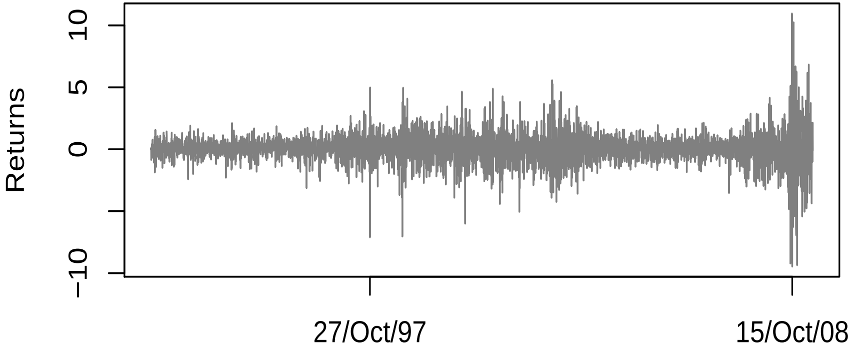


S&P 500, from March 2, 1992 to April 30, 2009

Return

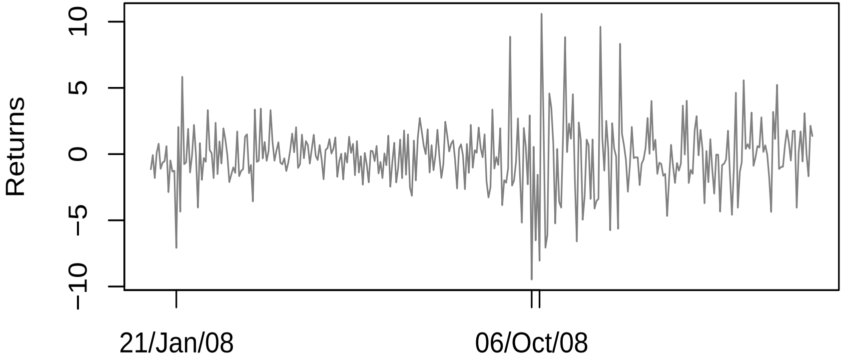
Stylized Facts

Possible stationarity of the returns



S&P 500 returns, from March 2, 1992 to April 30, 2009 Return

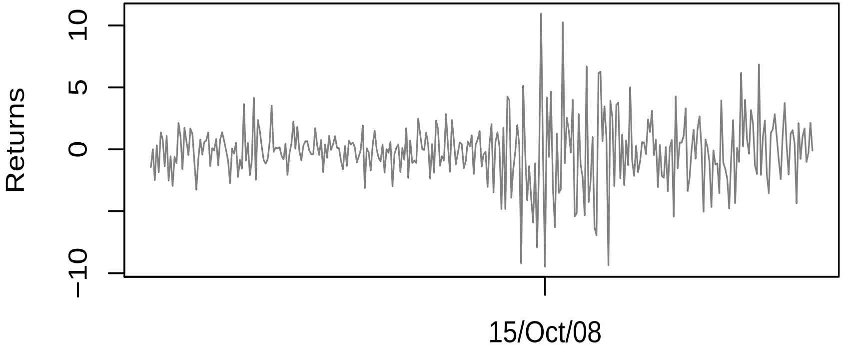
Stylized Facts
Volatility clustering



CAC 40 returns, from January 2, 2008 to April 30, 2009 ◂ Return

Stylized Facts

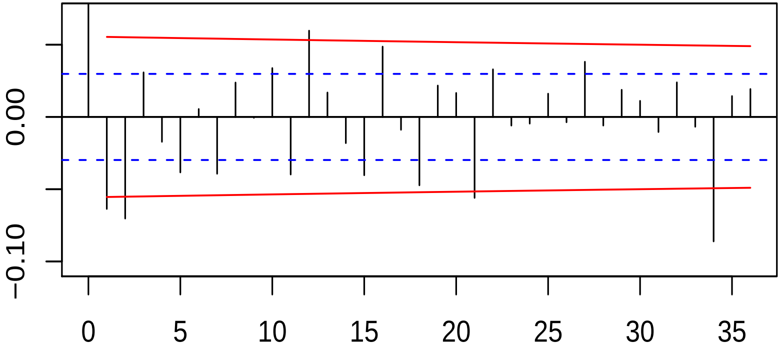
Conditional heteroskedasticity (compatible with marginal homoscedasticity and even stationarity)



S&P 500 returns, from January 2, 2008 to April 30, 2009 Return

Stylized Facts

Dependence without correlation (see FZ 2009 for the interpretation of the red lines)

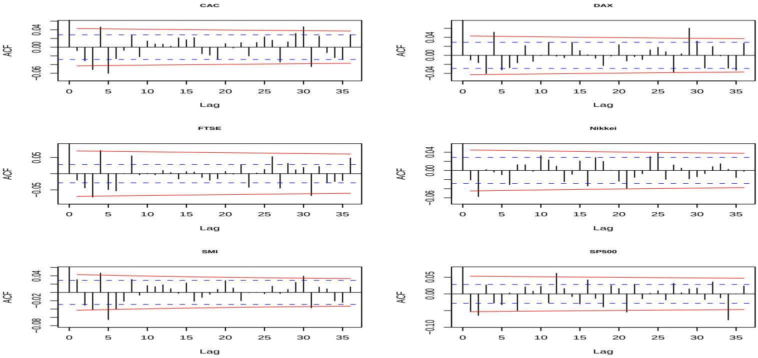


Empirical autocorrelations of the S&P 500 returns

Return

Stylized Facts

Dependence without correlation (significance bands under the GARCH(1,1) assumption)

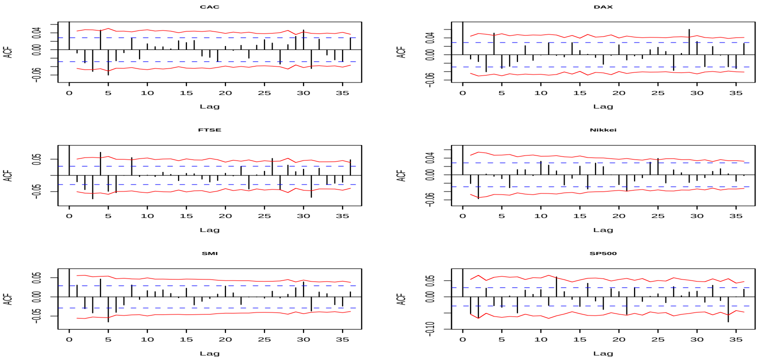


Empirical autocorrelations of daily stock returns

◂ Return

Stylized Facts

Dependence without correlation (the significance bands in red are estimated nonparametrically)

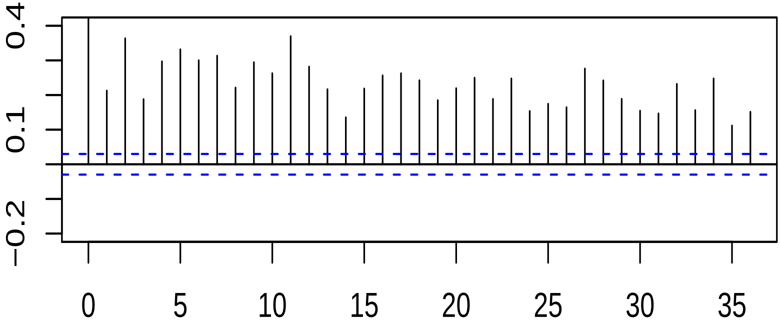


Empirical autocorrelations of daily stock returns

Return

Stylized Facts

Correlation of the squares

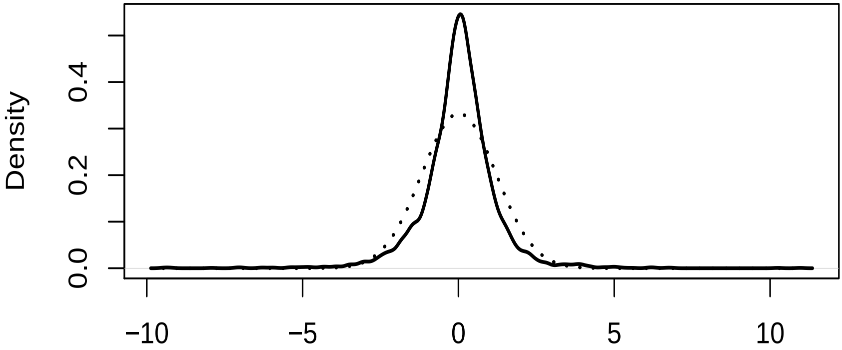


Autocorrelations of the squares of the S&P 500 returns

Return

Stylized Facts

Tail heaviness of the distributions



Density estimator for the S&P 500 returns (normal in dotted line)

Return

Stylized Facts

Decreases of prices have an higher impact on the future volatility than increases of the same magnitude

Table: Autocorrelations of tranformations of the CAC returns ϵ

h	1	2	3	4	5	6
$\hat{\rho}(\epsilon^{+}_{t-h}, \|\epsilon_t\|)$	0.03	0.07	0.07	0.08	0.08	0.12
$\hat{\rho}(-\epsilon^{-}_{t-h}, \|\epsilon_t\|)$	0.18	0.20	0.22	0.18	0.21	0.15

SP 500
Return

Stylized Facts

Decreases of prices have an higher impact on the future volatility than increases of the same magnitude

Table: Autocorrelations of tranformations of the S&P 500 returns ϵ

h	1	2	3	4	5	6
$\hat{\rho}_{\epsilon}(h)$	-0.06	-0.07	0.03	-0.02	-0.04	0.01
$\hat{\rho}_{\lvert\epsilon\rvert}(h)$	0.26	0.34	0.29	0.32	0.36	0.32
$\hat{\rho}(\epsilon_{t-h}^{+}, \lvert\epsilon_t\rvert)$	0.06	0.12	0.11	0.14	0.15	0.16
$\hat{\rho}(-\epsilon_{t-h}^{-}, \lvert\epsilon_t\rvert)$	0.25	0.28	0.23	0.24	0.28	0.23

Return

Idea of the proof in the ARCH(1) case

The QMLE minimizes $Q_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \frac{\sigma_t^2(\theta_0)\eta_t^2}{\sigma_t^2(\theta)} + \log \sigma_t^2(\theta)$ with $\sigma_t^2(\theta) = \omega + \alpha\epsilon_{t-1}^2$. Since $\epsilon_{t-1}^2 \to \infty$ a.s.,

$$\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \to \frac{\alpha_0}{\alpha},$$

and we have

$$Q_n(\theta) - Q_n(\theta_0) \to \frac{\alpha_0}{\alpha} - 1 + \log \frac{\alpha}{\alpha_0},$$

which is minimized at $\alpha = \alpha_0$.

Return

Asymptotic variance of the QMLE

When $\gamma_0 < 0$, the asymptotic variance is $(\kappa_\eta - 1)J^{-1}$ with

$$J = E_\infty\left(\frac{1}{h_t^2}\frac{\partial\sigma_t^2}{\partial\theta}\frac{\partial\sigma_t^2}{\partial\theta'}(\theta_0)\right).$$

When $\gamma_0 \geq 0$, the asymptotic variance is $(\kappa_\eta - 1)I^{-1}$ with

$$I = \begin{pmatrix} \frac{1}{\alpha_0^2} & \frac{\nu_1}{\alpha_0\beta_0(1-\nu_1)} \\ \frac{\nu_1}{\alpha_0\beta_0(1-\nu_1)} & \frac{(1+\nu_1)\nu_2}{\beta_0^2(1-\nu_1)(1-\nu_2)} \end{pmatrix} \quad \text{with } \nu_i = E\left(\frac{\beta_0}{\alpha_0\eta_1^2+\beta_0}\right)^i.$$

Return

(Initial) Motivations

- Complement the CAN results obtained by Jensen and Rahbek (2004, Econometrica and 2006, ET) for a **constrained** QML estimator.
- Correct the false impression that "GARCH models can be consistently estimated without any stationarity constraint."

Return

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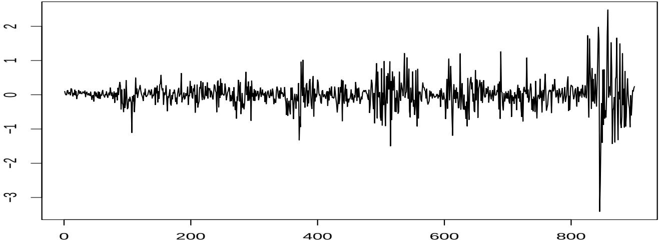
- Complement the CAN results obtained by Jensen and Rahbek (2004, Econometrica and 2006, ET) for a **constrained** QML estimator.
- Correct the false impression that "GARCH models can be consistently estimated without any stationarity constraint."

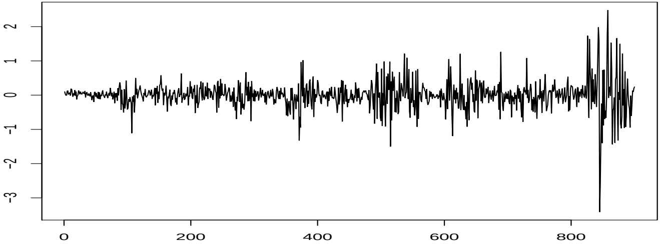
Return

GARCH Simulation



Is the simulated model stationary ?

GARCH Simulation

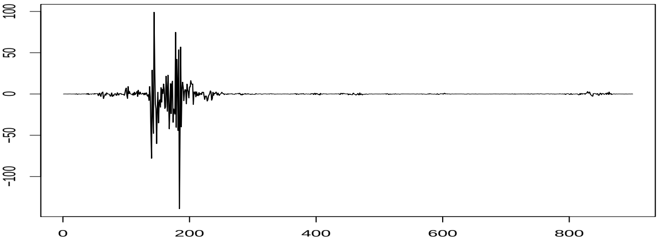


Yes: $\eta_t \sim St_7$ (standardized) $h_t = 0.001 + 0.2\epsilon_t^2 + 0.8h_{t-1}$

$\hat{\alpha}_n = 0.300, \quad \hat{\beta}_n = 0.746, \quad \hat{\gamma}_n = -3.44$ (p-val=0.9997)

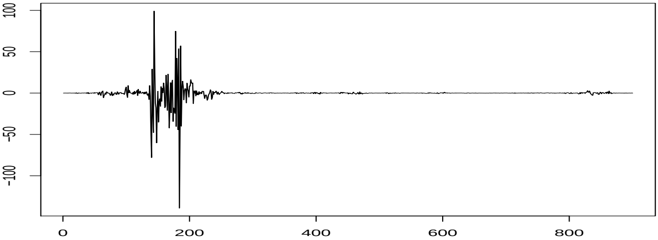
Return

GARCH Simulation



Is the simulated model stationary ?

GARCH Simulation

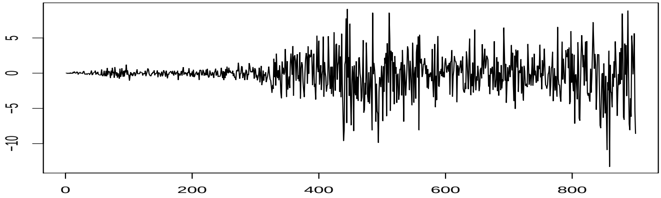


Yes: $\eta_t \sim St_5$ (standardized) $h_t = 0.001 + 0.93\epsilon_t^2 + 0.5h_{t-1}$

$\hat{\alpha}_n = 0.732, \quad \hat{\beta}_n = 0.504, \quad \hat{\gamma}_n = -3.01$ (p-val=0.9987)

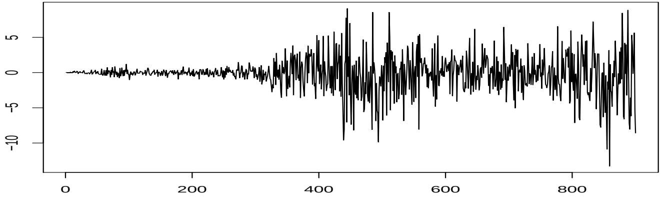
Return

GARCH Simulation



Is the simulated model stationary ?

GARCH Simulation



No: $\eta_t \sim \mathcal{N}(0,1)$ (standardized) $h_t = 0.001 + 0.12\epsilon_t^2 + 0.9h_{t-1}$

$$\hat{\alpha}_n = 0.080, \quad \hat{\beta}_n = 0.931, \quad \hat{\gamma}_n = 1.72 \text{ (p-val=0.042)}$$

Return

Score in the Explosive ARCH(1) Case

Since $\sigma_t^2(\theta) = \omega + \alpha\epsilon_{t-1}^2$ and $\epsilon_{t-1}^2 \to \infty$,

$$\frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \alpha} \to \frac{1}{\alpha},$$

and the score

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (1 - \eta_t^2) \frac{1}{\alpha_0} + o_P(1) \xrightarrow{d} \mathcal{N}\left(0, \frac{\kappa_\eta - 1}{\alpha_0^2}\right).$$