Estimating MGARCH models equation-by-equation

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Motivation and objectives

- The "dimensionality curse" is particularly problematic in multivariate GARCH models:
  - huge number of parameters;
  - inversion of the conditional variance matrix for the QMLE.
- Proposing an Equation-by-Equation Estimator (EbEE) for the volatility parameters of the individual components.
- Estimating the conditional correlations in a second step, based on the residuals of the EbEE.

Used in several empirical studies (see e.g. Sucarrat, Grønneberg and Escribano, 2013) without asymptotic results.

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General framework

Let $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \cdots, \epsilon_{mt})'$ and $\mathscr{F}_{t-1}^{X} = \sigma\{X_u, u < t\}$.
Assume $E(\boldsymbol{\epsilon}_t \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}) = 0$ and

$$\boldsymbol{H}_t = \mathrm{Var}(\boldsymbol{\epsilon}_t \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}) \text{ exists and is positive-definite.}$$

Let σ_{it}^2 denote the diagonal elements of $\boldsymbol{H}_t$ and let the equation by equation (EbE) innovations

$$\boldsymbol{\eta}_t^* = \boldsymbol{D}_t^{-1}\boldsymbol{\epsilon}_t = \begin{pmatrix} \frac{\epsilon_{1t}}{\sigma_{1t}} \\ \vdots \\ \frac{\epsilon_{mt}}{\sigma_{mt}} \end{pmatrix}, \quad \boldsymbol{D}_t = \mathrm{diag}(\sigma_{1t}, \ldots, \sigma_{mt}).$$

The conditional correlation matrix of $\boldsymbol{\epsilon}_t$ is given by

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"Semi-strong" DCC representation

Introducing the vector $\boldsymbol{\eta}_t$ such that $\boldsymbol{\eta}_t^* = \boldsymbol{R}_t^{1/2}\boldsymbol{\eta}_t$,

$$\begin{cases} \boldsymbol{\epsilon}_t &= \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \qquad E(\boldsymbol{\eta}_t \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}) = \boldsymbol{0}, \quad \mathrm{Var}(\boldsymbol{\eta}_t \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}) = \boldsymbol{I}_m, \\ \boldsymbol{H}_t &= \boldsymbol{H}(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \dots) = \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t, \end{cases}$$

where $\boldsymbol{D}_t = \{\mathrm{diag}(\boldsymbol{H}_t)\}^{1/2}$ and $\boldsymbol{R}_t = \mathrm{Corr}(\boldsymbol{\epsilon}_t \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}})$.

Remark: $(\boldsymbol{\eta}_t)$ is not an independent sequence in general (weak representation, that does not give the whole dynamics, but is quite general).

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Univariate "augmented" GARCH representations

Assuming that σ_{kt} has a parametric form, we then have

$$\begin{cases} \epsilon_{kt} & = & \sigma_{kt}\eta^*_{kt}, \\ \sigma_{kt} & = & \sigma_k(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \dots; \boldsymbol{\theta}_0^{(k)}), \end{cases}$$

where $\boldsymbol{\theta}_0^{(k)} \in \mathbb{R}^{d_k}$, $\sigma_k : \mathbb{R}^\infty \times \boldsymbol{\Theta}_k \to (0, \infty)$, and

$$E(\eta^*_{kt} \mid \mathscr{F}^{\boldsymbol{\epsilon}}_{t-1}) = 0, \qquad \text{Var}(\eta^*_{kt} \mid \mathscr{F}^{\boldsymbol{\epsilon}}_{t-1}) = 1.$$

Remarks:

- (η^*_{kt}) is not independent in general, as is usually assumed in GARCH modeling (not a DGP).
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DGP satisfying the semi-strong DCC representation

- GARCH-type models

$$\boldsymbol{\epsilon}_t = \boldsymbol{D}_t \boldsymbol{R}_t^{1/2} \boldsymbol{\eta}_t, \quad (\boldsymbol{\eta}_t) \text{ iid } (\boldsymbol{0}, \boldsymbol{I}_m).$$

  - Generalized Constant Conditional Correlation (CCC) models

$$\boldsymbol{R}_t = \boldsymbol{R} \text{ is a constant correlation matrix,}$$

  - Dynamic Conditional Correlation (DCC)

$$\boldsymbol{R}_t = \boldsymbol{R}(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \ldots) \neq \boldsymbol{R}.$$

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Stochastic Correlation Models

$$\boldsymbol{\epsilon}_t = \boldsymbol{D}_t \boldsymbol{R}_t^{*1/2} \boldsymbol{\xi}_t, \quad (\boldsymbol{\xi}_t) \text{ iid } (\boldsymbol{0}, \boldsymbol{I}_m)$$

$$\boldsymbol{R}_t^* = \boldsymbol{R}^*(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \dots, \Delta_t), \quad \Delta_t \notin \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$$

- Individual volatilities are of GARCH-type but correlations between components (in $\boldsymbol{R}_t^*$) are not
- If $\boldsymbol{\xi}_t$ is independent from $\mathscr{F}_t^{\Delta}$ and $\mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$,

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t, \quad \text{with} \quad \boldsymbol{R}_t = E(\boldsymbol{R}_t^* \mid \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}).$$

- Three innovations sequences

$$\boldsymbol{\eta}_t^* = \boldsymbol{R}_t^{*1/2} \boldsymbol{\xi}_t = \boldsymbol{R}_t^{1/2} \boldsymbol{\eta}_t.$$

Equation-by-equation gaussian QMLE (EbEE)

Given observations $\boldsymbol{\epsilon}_1,\dots,\boldsymbol{\epsilon}_n$, and arbitrary initial values $\tilde{\boldsymbol{\epsilon}}_i$ for $i \le 0$, a proxy of $\sigma_{kt}(\boldsymbol{\theta}^{(k)}) = \sigma_k(\boldsymbol{\epsilon}_{t-1},\boldsymbol{\epsilon}_{t-2},\dots,\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_0,\boldsymbol{\epsilon}_{-1},\dots;\boldsymbol{\theta}^{(k)})$ is defined by $\tilde{\sigma}_{kt}(\boldsymbol{\theta}^{(k)}) = \sigma_k(\boldsymbol{\epsilon}_{t-1},\boldsymbol{\epsilon}_{t-2},\dots,\boldsymbol{\epsilon}_1,\tilde{\boldsymbol{\epsilon}}_0,\tilde{\boldsymbol{\epsilon}}_{-1},\dots;\boldsymbol{\theta}^{(k)})$.

Define the EbEE of $(\boldsymbol{\theta}_0^{(1)},\dots,\boldsymbol{\theta}_0^{(m)})$ by, for $k=1,\dots,m$,

$$\hat{\boldsymbol{\theta}}_n^{(k)} = \arg\min_{\boldsymbol{\theta}^{(k)}\in\boldsymbol{\Theta}^{(k)}} \tilde{Q}_n^{(k)}(\boldsymbol{\theta}^{(k)}),$$

where

$$\tilde{Q}_n^{(k)}(\boldsymbol{\theta}^{(k)}) = \frac{1}{n}\sum_{t=1}^{n} \log\tilde{\sigma}_{kt}^2\left(\boldsymbol{\theta}^{(k)}\right) + \frac{\epsilon_{kt}^2}{\tilde{\sigma}_{kt}^2\left(\boldsymbol{\theta}^{(k)}\right)}.$$

Can we rely on the asymptotic theory of estimation for univariate GARCH? not in general because (η_{kt}^*) is not iid, and the volatility depends on all past components.

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CAN of the EbEE for augmented univariate models

$\epsilon_{kt} = \sigma_{kt}(\boldsymbol{\theta}_0^{(k)})\eta_{kt}^*$

Assume

- $(\boldsymbol{\epsilon}_t)$ is a strictly stationary and ergodic process, with $E|\epsilon_{kt}|^s < \infty$ for some $s > 0$, and $E \log \sigma_{kt}^2 < \infty$;
- $\boldsymbol{\theta}_0^{(k)}$ belongs to the interior of $\boldsymbol{\Theta}^{(k)}$;
- $E\left|\eta_{kt}^*\right|^{4(1+\delta)} < \infty$, for some $\delta > 0$,

and some additional technical assumptions, then

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$\hat{\boldsymbol{\theta}}_n^{(k)} \to \boldsymbol{\theta}_0^{(k)}, \quad a.s. \quad$ as $n \to \infty$ and
$\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n^{(k)} - \boldsymbol{\theta}_0^{(k)}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left\{0, \boldsymbol{J}_{kk}^{-1}\boldsymbol{I}_{kk}\boldsymbol{J}_{kk}^{-1}\right\}$, where $\boldsymbol{I}_{kk} = E\left(\{\eta_{kt}^{*4} - 1\}\boldsymbol{d}_{kt}\boldsymbol{d}_{kt}'\right)$,
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Asymptotic results for "quasi-strong" models

When

η_{kt}^* is independent from $\mathscr{F}_{t-1}^{\epsilon}$,

several assumptions can be weakened, for instance

$E\left|\eta_{kt}^*\right|^{4(1+\delta)} < \infty$ can be replaced by $E\left|\eta_{kt}^*\right|^4 < \infty$,

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- all the generalized CCC-GARCH models
- some DCC-GARCH and SC models. Condition for quasi-strong model

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Is the full QMLE always more efficient ?

Estimating the volatility coefficients EbE does not always entail efficiency loss with respect to the full QML.

Example: bivariate CCC model in which the only unknown coefficients are the parameters of the first volatility:

$$\boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \quad \boldsymbol{H}_t = \begin{pmatrix} \sigma_{1t}^2(\boldsymbol{\theta}_0^{(1)}) & \rho_0\sigma_{1t}(\boldsymbol{\theta}_0^{(1)})\sigma_{2t} \\ \rho_0\sigma_{1t}(\boldsymbol{\theta}_0^{(1)})\sigma_{2t} & \sigma_{2t}^2 \end{pmatrix}$$

The FQMLE of $\boldsymbol{\theta}_0^{(1)}$ is obtained by minimizing $\sum_{t=1}^n l_t(\boldsymbol{\theta}^{(1)})$ where

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where $\rho_0=\rho(\eta_{1t}^{*},\eta_{2t}^{*})$.

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GCCC model

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where $\boldsymbol{R}$ is a correlation matrix, $(\boldsymbol{\eta}_t)$ is an iid $(\boldsymbol{0},\boldsymbol{I}_m)$ process with $\boldsymbol{\eta}_t$ independent of $\mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$.
Let

$$\boldsymbol{\rho} = (R_{21},\dots,R_{m1},R_{32},\dots,R_{m2},\dots,R_{m,m-1})' = \mathrm{vech}^0(\boldsymbol{R}),$$

and the global parameter

$$\boldsymbol{\vartheta} = (\boldsymbol{\theta}^{(1)'},\dots,\boldsymbol{\theta}^{(m)'},\boldsymbol{\rho}')' := (\boldsymbol{\theta}',\boldsymbol{\rho}')' \in \mathbb{R}^d \times [-1,1]^{m(m-1)/2}, \quad d = \sum_{k=1}^m d_k.$$

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The two-step estimation procedure

1. EbEE of the $\boldsymbol{\theta}_0^{(k)}$'s and extraction of the residuals $\hat{\eta}_{kt}^*$;
2. Computation of the empirical correlation matrix

$$\hat{\boldsymbol{R}}_n = \frac{1}{n}\sum_{t=1}^{n} \hat{\boldsymbol{\eta}}_t^* \left(\hat{\boldsymbol{\eta}}_t^*\right)', \quad \hat{\boldsymbol{\eta}}_t^* = (\hat{\eta}_{1t}^*, \dots, \hat{\eta}_{mt}^*)'.$$

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CAN for Extended CCC models

Under some regularity conditions, as $n \to \infty$, $\hat{\boldsymbol{\vartheta}}_n \to \boldsymbol{\vartheta}_0$ a.s. and

$$\begin{pmatrix} \sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \\ \sqrt{n}(\hat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}_0) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left\{0, \boldsymbol{\Sigma} := \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\theta}} & \boldsymbol{\Sigma}_{\boldsymbol{\theta\rho}} \\ \boldsymbol{\Sigma}_{\boldsymbol{\theta\rho}}' & \boldsymbol{\Sigma}_{\boldsymbol{\rho}} \end{pmatrix}\right\}.$$

- Σ can be easily estimated by empirical means.
- Σ_θ is bloc-diagonal if $\mathrm{Cov}(\eta_{kt}^{*2}, \eta_{\ell t}^{*2}) = 0$ for any $k \neq \ell$.
- In general Σ_ρ depends on θ_0, but when $\boldsymbol{R} = \boldsymbol{I}_m$

$$\Sigma = \begin{pmatrix} \Sigma_\theta & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{m(m-1)/2} \end{pmatrix}, \qquad \Sigma_\theta = \mathrm{diag}((\kappa_{11}^* - 1)\boldsymbol{J}_{11}^{-1}, \dots, (\kappa_{mm}^* - 1)\boldsymbol{J}_{mm}^{-1}).$$

CAN of the two-step estimator

Let $\hat{\boldsymbol{\vartheta}}_n = \left(\hat{\boldsymbol{\theta}}_n' := (\hat{\boldsymbol{\theta}}_n^{(1)'}, \dots, \hat{\boldsymbol{\theta}}_n^{(m)'}), \hat{\boldsymbol{\rho}}_n'\right)'$, $\quad \hat{\boldsymbol{\rho}}_n = \text{vech}^0(\hat{\boldsymbol{R}}_n)$.

CAN for Extended CCC models

Under some regularity conditions, as $n \to \infty$, $\hat{\boldsymbol{\vartheta}}_n \to \boldsymbol{\vartheta}_0$ a.s. and

$$\begin{pmatrix} \sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \\ \sqrt{n}(\hat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}_0) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left\{0, \boldsymbol{\Sigma} := \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\theta}} & \boldsymbol{\Sigma}_{\boldsymbol{\theta}\boldsymbol{\rho}} \\ \boldsymbol{\Sigma}_{\boldsymbol{\theta}\boldsymbol{\rho}}' & \boldsymbol{\Sigma}_{\boldsymbol{\rho}} \end{pmatrix}\right\}.$$

- $\boldsymbol{\Sigma}$ can be easily estimated by empirical means.
- $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is bloc-diagonal if $\text{Cov}(\eta_{kt}^{*2}, \eta_{\ell t}^{*2}) = 0$ for any $k \neq \ell$.
- In general $\boldsymbol{\Sigma}_{\boldsymbol{\rho}}$ depends on $\boldsymbol{\theta}_0$, but when $\boldsymbol{R} = \boldsymbol{I}_m$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\theta}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{m(m-1)/2} \end{pmatrix}, \quad \boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \text{diag}((\kappa_{11}^* - 1)\boldsymbol{J}_{11}^{-1}, \dots, (\kappa_{mm}^* - 1)\boldsymbol{J}_{mm}^{-1}).$$

Time complexity comparison

CCC-GARCH$(1,1)$:

$$\boldsymbol{h}_t = \boldsymbol{\omega} + \boldsymbol{A}\underline{\boldsymbol{\epsilon}}_{t-1} + \boldsymbol{B}\boldsymbol{h}_{t-1}$$

where $\boldsymbol{h}_t = (\sigma^2_{1t}, \cdots, \sigma^2_{mt})'$, $\underline{\boldsymbol{\epsilon}}_t = (\epsilon^2_{1t}, \cdots, \epsilon^2_{mt})'$, $\boldsymbol{B}$ diagonal.

Conditional variance of the k-th component:

$$\sigma^2_{kt} = \omega_k + \sum_{j=1}^{m} \alpha_{kj}\epsilon^2_{j,t-1} + \beta_k\sigma^2_{k,t-1}.$$

Table: Number and dimension of the optimizations

Method	nb	dimension
EbEE	m	$m+2$
Full QMLE	1	$m^2+2m+m(m-1)/2$

Empirical comparison of the computation time

For time series of exchange rates of length $n = 2081$, using a single processor:

Table: CPU time in seconds

	dimension m				
	2	3	4	5	6
Estimator					
EbEE	15.59	28.50	43.91	70.90	98.39
FQMLE	101.41	443.34	870.04	1182.22	1515.58

A SC model

Assume $\boldsymbol{\epsilon}_t = \boldsymbol{D}_t \boldsymbol{R}_t^{*1/2} \boldsymbol{\xi}_t$,

$$\boldsymbol{R}_t^* = \boldsymbol{R}^*(\Delta_t) \text{ where } (\Delta_t) \text{ is a Markov chain on } \mathscr{E} = \{1, \dots, N\},$$

independent of $(\boldsymbol{\xi}_t)$. The Markov chain is not observed. Denoting by $p(i,j) = P(\Delta_t = j \mid \Delta_{t-1} = j)$ the transition probabilities, the parameter is

$$\begin{aligned}\boldsymbol{\zeta} &= (\boldsymbol{\theta}^{(1)'}, \dots, \boldsymbol{\theta}^{(m)'}, \boldsymbol{\rho}'(1), \dots, \boldsymbol{\rho}'(N), \boldsymbol{p}')' \\ &:= (\boldsymbol{\theta}', \boldsymbol{\rho}', \boldsymbol{p}')' \in \mathbb{R}^d \times [-1,1]^{Nm(m-1)/2} \times [0,1]^{N(N-1)},\end{aligned}$$

where $\boldsymbol{p} = (p(1,2), p(1,3), \dots, p(1,N), p(2,2), \dots, p(N,N))'$ and $\boldsymbol{\rho}(i) = \text{vech}^0\{\boldsymbol{R}(i)\}$ for $i = 1, \dots, N$.

Estimation of the SC model (without GARCH)

The HMM (Hidden Markov Model)

$$\boldsymbol{\eta}_t^* = \boldsymbol{R}_t^{*1/2}(\Delta_t)\boldsymbol{\xi}_t,$$

of unknown parameter $\boldsymbol{\vartheta}_0 = (\boldsymbol{\rho}_0', \boldsymbol{p}_0')'$ can be estimated by ML when $\boldsymbol{\eta}_1^*, \dots, \boldsymbol{\eta}_n^*$ are observed.
Assuming

- the sequences (Δ_t) and $(\boldsymbol{\xi}_t)$ are mutually independent,
- the Markov chain (Δ_t) is stationary, irreducible and aperiodic,
- $\boldsymbol{\xi}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_m)$,

and an identifiability assumption, the MLE of $\boldsymbol{\vartheta}_0$ is strongly consistent.

Adapting Hamilton's EM algorithm

Because the unknown parameters are correlations instead of covariances, the M step contains the non explicit maximization:

$$\boldsymbol{R}^*(i) = \arg\min_{\boldsymbol{R}\in\mathscr{R}} \log|\boldsymbol{R}| + \mathrm{Tr}\left\{\boldsymbol{R}^{-1}\boldsymbol{\Sigma}(i)\right\}$$

where $\mathscr{R}$ denotes the space of the $m \times m$ symmetric positive definite matrices and

$$\boldsymbol{\Sigma}(i) = \frac{1}{\sum_{t=1}^{n} \boldsymbol{\pi}_{t|n}(i)} \sum_{t=1}^{n} \boldsymbol{\eta}_t^*(\boldsymbol{\eta}_t^*)'\boldsymbol{\pi}_{t|n}(i).$$

Estimation of the SC-GARCH model

In practice, the innovations $\boldsymbol{\eta}_t^*$'s are not available. However

- The EbEE of $\boldsymbol{\theta}_0$ is consistent: $\hat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$ a.s.
- The EM algorithm can then be applied to the EbEE residuals

$$\hat{\boldsymbol{\eta}}_t^* = \tilde{\boldsymbol{D}}_t^{-1}(\hat{\boldsymbol{\theta}}_n)\boldsymbol{\epsilon}_t, \quad t = 1, \dots n.$$

Example of a bivariate BEKK-GARCH(1,1)

Consider for instance the simple model

$$\boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \qquad \boldsymbol{H}_t = \boldsymbol{\Omega} + \boldsymbol{A}\boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}'_{t-1}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{H}_{t-1},$$

where $(\boldsymbol{\eta}_t)$ iid $(\boldsymbol{0}, \boldsymbol{I}_2)$, $\boldsymbol{\Omega}$ and $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = \mathrm{diag}(b_1, b_2)$ are square 2×2 matrices, $\boldsymbol{\Omega}$ is positive definite, $b_1, b_2 \geq 0$.
The diagonal terms of $\boldsymbol{H}_t$ are given by

$$\begin{cases} h_{11,t} = \omega_{11} + a_{11}^2\epsilon_{1,t-1}^2 + 2a_{11}a_{12}\epsilon_{1,t-1}\epsilon_{2,t-1} + a_{12}^2\epsilon_{2,t-1}^2 + b_1 h_{11,t-1}, \\ h_{22,t} = \omega_{22} + a_{21}^2\epsilon_{1,t-1}^2 + 2a_{21}a_{22}\epsilon_{1,t-1}\epsilon_{2,t-1} + a_{22}^2\epsilon_{2,t-1}^2 + b_2 h_{22,t-1}. \end{cases}$$

Constraints on the augmented GARCH models

Letting $\boldsymbol{\theta}_0^{(k)} = (\omega_{kk}, a_{k1}^2, 2a_{k1}a_{k2}, a_{k2}^2)'$ for $k = 1,2$, the validity of this model can be studied by estimating, for $k = 1,2$,

$$\sigma_{kt}^2 = \theta_{01}^{(k)} + \theta_{02}^{(k)}\epsilon_{1,t-1}^2 + \theta_{03}^{(k)}\epsilon_{1,t-1}\epsilon_{2,t-1} + \theta_{04}^{(k)}\epsilon_{2,t-1}^2 + \theta_{05}^{(k)}\sigma_{k,t-1}^2,$$

under the positivity constraints $\theta_{01}^{(k)} > 0, \quad \theta_{0i}^{(k)} \geq 0, i = 2,5.$

The restrictions implied by the BEEK-GARCH(1,1) are:

H0(k): $\theta_{03}^{(k)} = 2\sqrt{\theta_{02}^{(k)}\theta_{04}^{(k)}}, \quad k = 1,2.$

Note that, under **H0(k)**, the true parameter value is at the boundary of the parameter set.

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Testing the BEKK formulation

Let the Wald statistic for the hypothesis **H0(k)**,

$$\boldsymbol{W}_n^{(k)} = \frac{n\left\{\hat{\theta}_{n3}^{(k)} - 2\sqrt{\hat{\theta}_{n2}^{(k)}\hat{\theta}_{n4}^{(k)}}\right\}^2}{\boldsymbol{X}_n'\hat{\boldsymbol{J}}_{kk}^{-1}\hat{\boldsymbol{I}}_{kk}\hat{\boldsymbol{J}}_{kk}^{-1}\boldsymbol{X}_n}, \qquad \text{where} \quad \hat{\boldsymbol{\theta}}_n^{(k)} = (\hat{\theta}_{n1}^{(k)}, \dots, \hat{\theta}_{n5}^{(k)})',$$

$\boldsymbol{X}_n = \left(0, \sqrt{\hat{\theta}_{n4}^{(k)}/\hat{\theta}_{n2}^{(k)}}, -1, \sqrt{\hat{\theta}_{n2}^{(k)}/\hat{\theta}_{n4}^{(k)}}, 0\right)'$, $\hat{\eta}_{kt}^{*} = \epsilon_{kt}/\tilde{\sigma}_{kt}(\hat{\boldsymbol{\theta}}_n^{(k)})$ and

$$\hat{\boldsymbol{J}}_{kk} = \frac{1}{n}\sum_{t=1}^{n}\hat{\boldsymbol{d}}_{kt}\hat{\boldsymbol{d}}_{kt}', \quad \hat{\boldsymbol{I}}_{kk} = \frac{1}{n}\sum_{t=1}^{n}\{\hat{\eta}_{kt}^{*4} - 1\}\hat{\boldsymbol{d}}_{kt}\hat{\boldsymbol{d}}_{kt}', \quad \hat{\boldsymbol{d}}_{kt} = \frac{1}{\tilde{\sigma}_{kt}^2(\hat{\boldsymbol{\theta}}_n)}\frac{\partial\tilde{\sigma}_{kt}^2(\hat{\boldsymbol{\theta}}_n^{(k)})}{\partial\boldsymbol{\theta}^{(k)}}.$$

Asymptotic distribution under the null

Suppose $\rho(\boldsymbol{A}+\boldsymbol{B})<1$ and let $a_{11}a_{12}>0$, $a_{21}a_{22}>0$. Suppose $\boldsymbol{\eta}_1$ admits a positive density around $\mathbf{0}$, and suppose that $E\left|\eta_{kt}\right|^{4(1+\delta)}<\infty$, for $k=1,2$ and some $\delta>0$. Then,

$$\boldsymbol{W}_n^{(k)} \xrightarrow{d} \frac{1}{2}\chi^2(1)+\frac{1}{2}\delta_0.$$

Testing **H0(k)** at the asymptotic level $\alpha \in (0,1/2)$ can thus be achieved by using the critical region $\{\boldsymbol{W}_n^{(k)} > \chi^2_{1-2\alpha}(1)\}$.

CCC-GARCH(1,1) $\underline{h}_t = \underline{\omega} + \mathbf{A}\underline{\epsilon}_{t-1} + \mathbf{B}\underline{h}_{t-1}$, $\mathbf{B}$ diagonal

EbEE

$$\hat{\mathbf{A}} = \begin{array}{c} \left(\begin{array}{cccccc} \underset{0.010}{0.029} & \underset{0.003}{0.002} & \underset{0.040}{0.015} & \underset{0.013}{0.012} & \underset{0.003}{0.003} & \underset{0.038}{0.000} \\ \underset{0.002}{0.000} & \underset{0.023}{0.136} & \underset{0.004}{0.000} & \underset{0.003}{0.003} & \underset{0.001}{0.000} & \underset{0.003}{0.000} \\ \underset{0.005}{0.000} & \underset{0.002}{0.002} & \underset{0.028}{0.031} & \underset{0.007}{0.008} & \underset{0.002}{0.002} & \underset{0.027}{0.001} \\ \underset{0.004}{0.006} & \underset{0.002}{0.001} & \underset{0.020}{0.004} & \underset{0.012}{0.041} & \underset{0.002}{0.006} & \underset{0.019}{0.000} \\ \underset{0.012}{0.017} & \underset{0.005}{0.003} & \underset{0.054}{0.000} & \underset{0.016}{0.002} & \underset{0.012}{0.061} & \underset{0.052}{0.000} \\ \underset{0.005}{0.000} & \underset{0.002}{0.003} & \underset{0.028}{0.024} & \underset{0.007}{0.007} & \underset{0.002}{0.002} & \underset{0.028}{0.008} \end{array} \right), \end{array} \begin{array}{c} \text{CAD} \\ \text{CHF} \\ \text{CNY} \\ \text{GBP} \\ \text{JPY} \\ \text{USD} \end{array}$$

Outside the diagonal, the coefficients are not significant.

$$(\text{diag}\hat{\mathbf{B}})' = \left(\begin{array}{cccccc} \underset{0.022}{0.92} & \underset{0.017}{0.88} & \underset{0.010}{0.95} & \underset{0.015}{0.93} & \underset{0.014}{0.93} & \underset{0.009}{0.96} \end{array} \right).$$

Correlation matrix of the CCC-GARCH(1,1) model

Second step estimator

$$\hat{R} = \begin{pmatrix} 1.00 & \underset{0.026}{0.00} & \underset{0.039}{0.46} & \underset{0.031}{0.39} & \underset{0.034}{0.17} & \underset{0.032}{0.47} \\ 0.00 & 1.00 & \underset{0.040}{0.14} & \underset{0.027}{0.12} & \underset{0.043}{0.42} & \underset{0.045}{0.13} \\ 0.46 & 0.14 & 1.00 & \underset{0.033}{0.44} & \underset{0.039}{0.58} & \underset{0.031}{0.98} \\ 0.39 & 0.12 & 0.44 & 1.00 & \underset{0.071}{0.26} & \underset{0.040}{0.45} \\ 0.17 & 0.42 & 0.58 & 0.26 & 1.00 & \underset{0.044}{0.57} \\ 0.47 & 0.13 & 0.98 & 0.45 & 0.57 & 1.00 \end{pmatrix} \begin{array}{l} \text{CAD} \\ \text{CHF} \\ \text{CNY} \\ \text{GBP} \\ \text{JPY} \\ \text{USD} \end{array}$$

Instantaneous positive correlations.

MS correlation matrix with 2 regimes

Second step estimator of the fist regime

The EbEE of the first step remains the same.

$$
\hat{\boldsymbol{R}}(1) = \left(\begin{array}{cccccc}
1.00 & \underset{0.150}{0.38} & \underset{0.062}{0.71} & \underset{0.141}{0.69} & \underset{0.127}{0.58} & \underset{0.061}{0.72} \\
0.38 & 1.00 & \underset{0.138}{0.59} & \underset{0.107}{0.52} & \underset{0.066}{0.66} & \underset{0.140}{0.59} \\
0.71 & 0.59 & 1.00 & \underset{0.132}{0.81} & \underset{0.096}{0.89} & \underset{0.002}{0.99} \\
0.69 & 0.52 & 0.81 & 1.00 & \underset{0.146}{0.76} & \underset{0.135}{0.82} \\
0.58 & 0.66 & 0.89 & 0.76 & 1.00 & \underset{0.101}{0.90} \\
0.72 & 0.59 & 0.99 & 0.82 & 0.90 & 1.00
\end{array}\right)
\begin{array}{l}
\text{CAD} \\ \text{CHF} \\ \text{CNY} \\ \text{GBP} \\ \text{JPY} \\ \text{USD}
\end{array}
$$

MS correlation matrix with 2 regimes

Second step estimator of the second regime

Two different regimes for the correlations.

$$\hat{\boldsymbol{R}}(2) = \begin{array}{cl} \left(\begin{array}{cccccc} 1.00 & \underset{0.039}{-0.04} & \underset{0.029}{0.42} & \underset{0.030}{0.34} & \underset{0.042}{0.10} & \underset{0.028}{0.43} \\ -0.04 & 1.00 & \underset{0.044}{0.08} & \underset{0.039}{0.08} & \underset{0.028}{0.39} & \underset{0.044}{0.07} \\ 0.42 & 0.08 & 1.00 & \underset{0.039}{0.38} & \underset{0.033}{0.52} & \underset{0.001}{0.98} \\ 0.34 & 0.08 & 0.38 & 1.00 & \underset{0.051}{0.18} & \underset{0.039}{0.38} \\ 0.10 & 0.39 & 0.52 & 0.18 & 1.00 & \underset{0.034}{0.51} \\ 0.43 & 0.07 & 0.98 & 0.38 & 0.51 & 1.00 \end{array}\right) & \begin{array}{l} \text{CAD} \\ \text{CHF} \\ \text{CNY} \\ \text{GBP} \\ \text{JPY} \\ \text{USD} \end{array} \end{array}$$

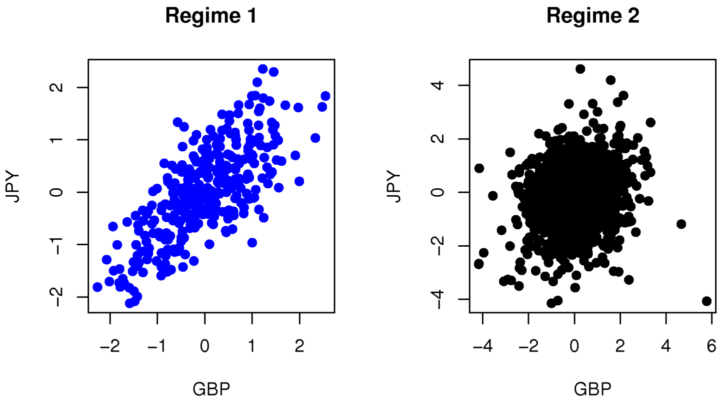
MS correlation matrix with 2 regimes

Estimated transition probabilities

$$\hat{\boldsymbol{P}} = \begin{pmatrix} \underset{0.036}{0.826} & \underset{0.036}{0.174} \\ \underset{0.013}{0.039} & \underset{0.013}{0.961} \end{pmatrix}.$$

This corresponds to regimes with relative frequencies $\hat{P}(\Delta_t = 1) = 0.18$ and $\hat{P}(\Delta_t = 2) = 0.82$.

GBP and JPY residuals as function of the most probable regime



For each pair of exchange rates:

p-values of the tests of the null hypotheses $H_0^{(1)}$ and $H_0^{(2)}$ implied by the bivariate BEKK-GARCH(1,1) model.

	CAD		CHF		CNY		GBP		JPY	
	$H_0^{(1)}$	$H_0^{(2)}$	$H_0^{(1)}$	$H_0^{(2)}$	$H_0^{(1)}$	$H_0^{(2)}$	$H_0^{(1)}$	$H_0^{(2)}$	$H_0^{(1)}$	$H_0^{(2)}$
CHF	0.000	0.163								
CNY	0.120	0.015	0.122	0.500						
GBP	0.012	0.023	0.128	0.000	0.005	0.100				
JPY	0.007	0.006	0.500	0.500	0.500	0.087	0.050	0.000		
USD	0.500	0.021	0.114	0.000	0.500	0.381	0.068	0.000	0.102	0.000

25 world stock market indices

- Major world stock market indices by Yahoo: 5 for Americas, 11 for Asia-Pacific, 8 for Europe and 1 for Middle East;
- from $n = 2157$ for the series "NZ50" to $n = 6040$ for "AEX.AS" (1990 to mid 2013);
- CCC model with individual PGARCH(1,1) volatilities

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t^\delta = \omega + \alpha_+ (\epsilon_{t-1}^+)^\delta + \alpha_- (-\epsilon_{t-1}^-)^\delta + \beta \sigma_{t-1}^\delta \end{cases}$$

with $\delta \in \{0.5, 1, 1.5, 2\}$.

PGARCH(1,1) models of the major World stock indices

	$\widehat{\omega}$	$\widehat{\alpha}_+$	$\widehat{\alpha}_-$	$\widehat{\beta}$	$\widehat{\delta}$
MERV	0.151 (0.002)	0.063 (0.002)	0.151 (0.001)	0.858 (0.004)	2
BVSP	0.077 (0.001)	0.068 (0.001)	0.138 (0.002)	0.884 (0.002)	2
GSPT	0.012 (0.009)	0.046 (0.002)	0.109 (0.004)	0.926 (0.007)	1
MXX	0.032 (0.003)	0.044 (0.001)	0.167 (0.002)	0.896 (0.004)	1.5
GSPC	0.016 (0.006)	0.000 (0.002)	0.134 (0.003)	0.927 (0.004)	1.5
AORD	0.023 (0.007)	0.030 (0.002)	0.131 (0.003)	0.910 (0.006)	1
SSEC	0.031 (0.010)	0.082 (0.004)	0.123 (0.003)	0.904 (0.012)	1
HSI	0.029 (0.008)	0.049 (0.003)	0.120 (0.003)	0.916 (0.009)	1
BSES	0.055 (0.004)	0.062 (0.003)	0.179 (0.002)	0.872 (0.005)	1.5
JKSE	0.063 (0.005)	0.096 (0.002)	0.190 (0.001)	0.856 (0.005)	1.5
KLSE	0.087 (0.022)	0.071 (0.002)	0.157 (0.001)	0.835 (0.014)	2
N225	0.044 (0.004)	0.038 (0.003)	0.148 (0.002)	0.898 (0.006)	1
NZ50	0.018 (0.019)	0.044 (0.006)	0.120 (0.004)	0.898 (0.010)	1.5
STI	0.027 (0.011)	0.078 (0.001)	0.178 (0.001)	0.876 (0.005)	1.5
KS11	0.017 (0.009)	0.049 (0.001)	0.121 (0.004)	0.923 (0.008)	1.5
TWII	0.028 (0.012)	0.041 (0.004)	0.123 (0.003)	0.918 (0.010)	1
ATX	0.030 (0.005)	0.050 (0.002)	0.137 (0.003)	0.902 (0.007)	1

Correlation matrix estimate $\hat{R}$ (pairwise method)

	MER	BVS	GST	MXX	GSC	AOR	SSE	HSI	BSE	JKS	KLS	N22	NZ5
MERV	1.00												
BVSP	0.53	1.00											
GSPT	0.47	0.48	1.00										
MXX	0.47	0.52	0.48	1.00									
GSPC	0.48	0.52	0.67	0.55	1.00								
AORD	0.17	0.17	0.21	0.17	0.12	1.00							
SSEC	0.06	0.08	0.08	0.06	0.02	0.18	1.00						
HSI	0.21	0.19	0.22	0.21	0.14	0.49	0.28	1.00					
BSES	0.17	0.19	0.21	0.20	0.15	0.31	0.14	0.40	1.00				
JKSE	0.15	0.15	0.14	0.15	0.08	0.36	0.15	0.43	0.31	1.00			
KLSE	0.10	0.10	0.11	0.12	0.06	0.28	0.14	0.36	0.19	0.32	1.00		
N225	0.11	0.13	0.19	0.12	0.12	0.46	0.16	0.44	0.27	0.34	0.28	1.00	
NZ50	0.09	0.06	0.10	0.09	0.04	0.48	0.16	0.31	0.21	0.29	0.22	0.38	1.00
STI	0.22	0.20	0.22	0.20	0.16	0.44	0.18	0.56	0.38	0.44	0.39	0.40	0.32
KS11	0.15	0.20	0.20	0.20	0.15	0.49	0.16	0.55	0.33	0.36	0.27	0.54	0.32
TWII	0.13	0.14	0.15	0.13	0.10	0.41	0.18	0.47	0.27	0.33	0.27	0.44	0.31
ATX	0.31	0.27	0.33	0.30	0.30	0.32	0.12	0.33	0.27	0.28	0.19	0.27	0.22
BFX	0.35	0.33	0.40	0.36	0.42	0.30	0.09	0.31	0.27	0.24	0.17	0.25	0.20
FCHI	0.37	0.36	0.44	0.39	0.47	0.26	0.06	0.31	0.28	0.21	0.15	0.26	0.17
GDAX	0.36	0.37	0.44	0.38	0.47	0.30	0.07	0.34	0.28	0.21	0.16	0.27	0.16
AEX	0.37	0.36	0.45	0.39	0.45	0.31	0.06	0.35	0.29	0.22	0.18	0.28	0.18
SSMI	0.33	0.31	0.39	0.35	0.41	0.29	0.05	0.31	0.27	0.23	0.16	0.27	0.19
FTSE	0.38	0.37	0.46	0.39	0.47	0.28	0.06	0.32	0.29	0.22	0.17	0.27	0.18

Correlation matrix estimate $\hat{R}$

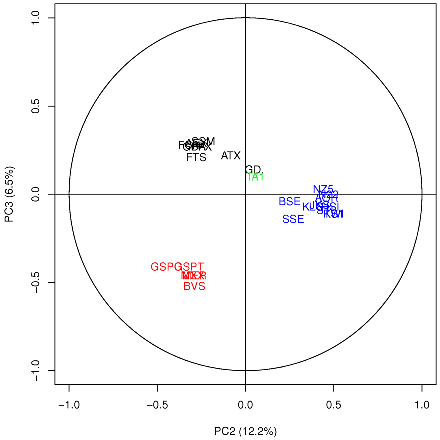
	STI	KS1	TWI	ATX	BFX	FCH	GDA	AEX	SSM	FTS	GD	TA1
STI	1.00											
KS11	0.50	1.00										
TWII	0.45	0.51	1.00									
ATX	0.32	0.28	0.23	1.00								
BFX	0.30	0.25	0.19	0.56	1.00							
FCHI	0.30	0.26	0.20	0.55	0.71	1.00						
GDAX	0.31	0.27	0.20	0.59	0.70	0.79	1.00					
AEX	0.33	0.28	0.22	0.58	0.74	0.82	0.79	1.00				
SSMI	0.30	0.26	0.21	0.52	0.66	0.72	0.72	0.74	1.00			
FTSE	0.31	0.27	0.19	0.54	0.66	0.77	0.70	0.76	0.69	1.00		
GD	0.25	0.27	0.21	0.32	0.34	0.34	0.33	0.33	0.32	0.30	1.00	
TA10	0.36	0.28	0.25	0.38	0.39	0.42	0.40	0.41	0.40	0.40	0.33	1.00

PCA loading matrix: correlations between the variables and the first 3 factors

	PC1	PC2	PC3
MERV	-0.52	-0.29	-0.46
BVSP	-0.52	-0.29	-0.52
GSPT	-0.59	-0.32	-0.41
MXX	-0.54	-0.30	-0.46
GSPC	-0.56	-0.45	-0.41
AORD	-0.55	0.46	-0.02
SSEC	-0.19	0.27	-0.14
HSI	-0.60	0.48	-0.07
BSES	-0.48	0.25	-0.04
JKSE	-0.45	0.42	-0.06
KLSE	-0.35	0.38	-0.07
N225	-0.50	0.47	-0.00
NZ50	-0.37	0.44	0.03

	PC1	PC2	PC3
STI	-0.58	0.45	-0.09
KS11	-0.55	0.50	-0.11
TWII	-0.46	0.50	-0.11
ATX	-0.68	-0.08	0.22
BFX	-0.75	-0.25	0.27
FCH	-0.79	-0.32	0.28
GDA	-0.79	-0.29	0.27
AEX	-0.81	-0.28	0.29
SSM	-0.75	-0.24	0.30
FTS	-0.78	-0.28	0.21
GD.	-0.46	0.05	0.14
TA10	-0.57	0.06	0.10
	34.6%	12.2%	6.5%

Factorial plan PC2-PC3



Conclusion

EbEE + correlation of the EbE residuals

- much simpler than the FQMLE;
- not necessarily less efficient;
- first EbEE step valid for different correlation structures;
- specification tests;
- asynchronous individual series.

Preprint: `http://mpra.ub.uni-muenchen.de/54250/`

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Thanks for your attention ☺ !

DCC-GARCH: A condition to obtain a quasi-strong model

$$\boldsymbol{\epsilon}_t = \boldsymbol{D}_t \boldsymbol{\eta}_t^*, \quad \boldsymbol{\eta}_t^* = \boldsymbol{R}_t^{1/2} \boldsymbol{\eta}_t, \quad (\boldsymbol{\eta}_t) \text{ iid } (\boldsymbol{0}, \boldsymbol{I}_m), \quad \boldsymbol{R}_t \in \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$$

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Proof. Denoting by $\boldsymbol{e}_k$ the k-th column of $\boldsymbol{I}_m$, we have

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conditionally to $\mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$, and thus unconditionally.

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Remark: the process $(\boldsymbol{\eta}_t^*)$ is not independent in general:

$$\lambda_1 \eta_{kt}^* + \lambda_2 \eta_{\ell t}^* \stackrel{d}{=} \|(\lambda_1 \boldsymbol{e}_k' + \lambda_2 \boldsymbol{e}_\ell') \boldsymbol{R}_t^{1/2}\| \eta_1 = \{\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \boldsymbol{R}_t(k,\ell)\}^{1/2} \eta_1,$$

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- If in addition $\mathscr{F}_{t-1}^{\boldsymbol{\epsilon}} = \mathscr{F}_{t-1}^{\boldsymbol{\eta}^*}$ then the augmented univariate GARCH representations are quasi-strong (the asymptotic variance is then simpler).
- The multivariate model is not strong in general, since the $\boldsymbol{\eta}_t^*$'s are not independent, and not id (when $(\boldsymbol{R}_t)$ is not stationary).

Information sets: condition for $\mathscr{F}_{t-1}^{\boldsymbol{\epsilon}} = \mathscr{F}_{t-1}^{\boldsymbol{\eta}^*}$

$\boldsymbol{\epsilon}_t = \boldsymbol{D}_t \boldsymbol{\eta}_t^*$ entails $\mathscr{F}_{t-1}^{\boldsymbol{\eta}^*} \subset \mathscr{F}_{t-1}^{\boldsymbol{\epsilon}}$. For some models (but not all),

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Example: Multivariate ARCH(1) model with

$$\sigma_{it}^2 = \omega_i + \sum_{j=1}^{m} \alpha_{ij} \epsilon_{j,t-1}^2.$$

Letting $\underline{h}_t = (\sigma_{1t}^2, \ldots, \sigma_{mt}^2)'$ and $\underline{\omega} = (\omega_1, \ldots, \omega_m)'$, we have

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where $\mathbf{A}(\boldsymbol{\eta}_{t-1}^*) = (\alpha_{ij}\eta_{j,t-1}^{*2})_{i,j}$. It follows that

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Return

Proof that η_{kt}^* is independent of $\boldsymbol{\eta}_{t-1}^* = \boldsymbol{R}_{t-1}^{*1/2}\boldsymbol{\xi}_{t-1}$

Using the independence between $\boldsymbol{\xi}_t$ and $\boldsymbol{\xi}_{t-1}$ and between $(\boldsymbol{R}_t^*)$ and $(\boldsymbol{\xi}_t)$,

$$P(\eta_{kt}^* < x, \eta_{\ell,t-1}^* < y \mid \boldsymbol{R}_t^*, \boldsymbol{R}_{t-1}^*) \quad = \quad P(\eta_{kt}^* < x \mid \boldsymbol{R}_t^*) P(\eta_{\ell,t-1}^* < y \mid \boldsymbol{R}_{t-1}^*).$$

Because (conditional to $\boldsymbol{R}_t^*$), $\boldsymbol{\xi}_t$ is spherically distributed,

$$\eta_{kt}^* = \boldsymbol{e}_k' \boldsymbol{R}_t^{*1/2} \boldsymbol{\xi}_t \stackrel{d}{=} \|\boldsymbol{e}_k' \boldsymbol{R}_t^{*1/2}\| \xi_1 = \xi_1$$

and thus

$$P(\eta_{kt}^* < x, \eta_{\ell,t-1}^* < y \mid \boldsymbol{R}_t^*, \boldsymbol{R}_{t-1}^*) = P(\eta_{kt}^* < x) P(\eta_{\ell,t-1}^* < y).$$

Return

Technical assumptions for the consistency of the EbEE

- for any real sequence $(e_i)_{i\geq 1}$, the function $\boldsymbol{\theta}^{(k)} \mapsto \sigma_k(e_1,e_2,\dots;\boldsymbol{\theta}^{(k)})$ is continuous and there exists $K:\mathbb{R}^\infty \mapsto (0,\infty)$ such that

$$|\sigma_k(e_1,e_2,\dots;\boldsymbol{\theta}^{(k)}) - \sigma_k(e_1,e_2,\dots;\boldsymbol{\theta}_0^{(k)})| \leq K(e_1,\dots)\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}_0^{(k)}\|,$$

$$E\left(\frac{K(\boldsymbol{\epsilon}_{t-1},\boldsymbol{\epsilon}_{t-2},\dots)}{\sigma_{kt}(\boldsymbol{\theta}_0^{(k)})}\right)^2 < \infty.$$

- there exists a neighborhood $\mathcal{V}(\boldsymbol{\theta}_0^{(k)})$ of $\boldsymbol{\theta}_0^{(k)}$ such that

$$E \sup_{\boldsymbol{\theta}^{(k)} \in \mathcal{V}(\boldsymbol{\theta}_0^{(k)})} \left(\frac{\sigma_{kt}(\boldsymbol{\theta}_0^{(k)})}{\sigma_{kt}(\boldsymbol{\theta}^{(k)})}\right)^2 < \infty.$$

- $\sigma_{kt}(\cdot) > \underline{\omega}$ for some $\underline{\omega} > 0$.
- $\sigma_{kt}(\boldsymbol{\theta}_0^{(k)}) = \sigma_{kt}(\boldsymbol{\theta}^{(k)})\ a.s.$ iff $\boldsymbol{\theta}^{(k)} = \boldsymbol{\theta}_0^{(k)}$.
- Let $\Delta_{kt} = \tilde{\sigma}_{kt}(\boldsymbol{\theta}^{(k)}) - \sigma_{kt}(\boldsymbol{\theta}^{(k)})$. Let $C > 0$ and $0 < \rho < 1$. We have

$$\sup_{\boldsymbol{\theta}^{(k)} \in \boldsymbol{\Theta}^{(k)}} |\Delta_{kt}| \leq C\rho^t, \qquad a.s.$$

Technical assumptions for the AN of the EbEE

- for any real sequence $(e_i)_{i\geq 1}$, the function $\boldsymbol{\theta}^{(k)} \mapsto \sigma_k(e_1, e_2, \dots; \boldsymbol{\theta}^{(k)})$ has continuous second-order derivatives;
- there exists a neighborhood $\mathcal{V}(\boldsymbol{\theta}_0^{(k)})$ of $\boldsymbol{\theta}_0^{(k)}$ such that

$$\sup_{\boldsymbol{\theta}^{(k)}\in\mathcal{V}(\boldsymbol{\theta}_0^{(k)})} \left\| \frac{1}{\sigma_{kt}(\boldsymbol{\theta}^{(k)})} \frac{\partial \sigma_{kt}(\boldsymbol{\theta}^{(k)})}{\partial \boldsymbol{\theta}^{(k)}} \right\|^{4(1+\frac{1}{\delta})}, \quad \sup_{\boldsymbol{\theta}^{(k)}\in\mathcal{V}(\boldsymbol{\theta}_0^{(k)})} \left\| \frac{1}{\sigma_{kt}(\boldsymbol{\theta}^{(k)})} \frac{\partial^2 \sigma_{kt}(\boldsymbol{\theta}^{(k)})}{\partial \boldsymbol{\theta}^{(k)} \partial \boldsymbol{\theta}^{(k)'}} \right\|^{2(1+\frac{1}{\delta})},$$

$$\sup_{\boldsymbol{\theta}^{(k)}\in\mathcal{V}(\boldsymbol{\theta}_0^{(k)})} \left| \frac{\sigma_{kt}(\boldsymbol{\theta}_0^{(k)})}{\sigma_{kt}(\boldsymbol{\theta}^{(k)})} \right|^4, \quad \text{have finite expectations.}$$

-

$$\sup_{\boldsymbol{\theta}^{(k)}\in\boldsymbol{\Theta}^{(k)}} \left\| \frac{\partial \Delta_{kt}(\boldsymbol{\theta}^{(k)})}{\partial \boldsymbol{\theta}^{(k)}} \right\| \leq C\rho^t, \qquad a.s.$$

- For $k = 1, \dots, m$ and for any $\boldsymbol{x} \in \mathbb{R}^{d_k}$,

$$\boldsymbol{x}' \frac{\partial \sigma_{kt}^2(\boldsymbol{\theta}_0^{(k)})}{\partial \boldsymbol{\theta}^{(k)}} = 0, \; a.s. \quad \Rightarrow \quad \boldsymbol{x} = 0.$$

Return