Testing the nullity of GARCH coefficients:

correction of the standard tests and relative efficiency comparisons

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Outline

1. QMLE of GARCH models
2. Test hypotheses and statistics
3. Testing the nullity of one coefficient
4. Testing conditional homoskedasticity versus ARCH(q)
5. Financial application and conclusion

Definition: GARCH(p,q)

Engle (1982), Bollerslev (1986)

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t^2 = \omega_0 + \sum_{i=1}^q \alpha_{0i}\epsilon_{t-i}^2 + \sum_{j=1}^p \beta_{0j}\sigma_{t-j}^2, \quad \forall t \in \mathbb{Z} \end{cases}$$

(η_t) iid, $E\eta_t = 0$, $E\eta_t^2 = 1$,

$\omega_0 > 0 \quad , \alpha_{0i} \geq 0 \ (i = 1, \dots, q) \quad , \beta_{0j} \geq 0 \ (j = 1, \dots, p)$.

$$\theta_0 = (\omega_0, \alpha_{01}, \dots, \alpha_{0q}, \beta_{01}, \dots, \beta_{0p}).$$

Stricty stationarity

$$A_{0t} = \left(\begin{array}{cccccc} \alpha_{01}\eta_t^2 & \cdots & \alpha_{0q}\eta_t^2 & \beta_{01}\eta_t^2 & \cdots & \beta_{0p}\eta_t^2 \\ & I_{q-1} & 0 & & 0 & \\ \alpha_{01} & \cdots & \alpha_{0q} & \beta_{01} & \cdots & \beta_{0p} \\ & 0 & & & I_{p-1} & 0 \end{array} \right).$$

$$\gamma(\mathbf{A_0}) \quad = \quad \lim_{t\to\infty} a.s. \, \frac{1}{t} \log \| A_{0t} A_{0t-1} \ldots A_{01} \|.$$

Theorem

The model has a (unique) strictly stationary non anticipative solution iff

$$\gamma(\mathbf{A_0}) < 0.$$

[Bougerol & Picard, 1992]

Quasi-Maximum Likelihood Estimation

A QMLE of θ is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} \tilde{\mathbf{l}}_n(\theta),$$

where $\tilde{\mathbf{l}}_n(\theta) = n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t,$ and $\tilde{\ell}_t = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2} + \log \tilde{\sigma}_t^2.$

Remarks:

- The constraint $\tilde{\sigma}_t^2 > 0$ for all $\theta \in \Theta$ is necessary to compute $\tilde{\mathbf{l}}_n(\theta)$.
- The QMLE is always constrained: the "unrestricted" QMLE does not exist.

Quasi-Maximum Likelihood Estimation

Under appropriate conditions [in particular strict stationarity and $\theta_0 > 0$] (Berkes, Horváth and Kokoszka (2003), FZ (2004))

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (\kappa_\eta - 1)J^{-1}),$$

$$\kappa_\eta = E\eta_t^4, \quad J = E_{\theta_0}\left(\frac{1}{\sigma_t^4(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta'}\right).$$

Remark: The strict stationarity condition is essential:

- Without strict stationarity, it is possible to consistently estimate α in an ARCH(1) (Jensen and Rahbeck, 2004), but not the intercept ω.
- When the process is not strictly stationary, $\sigma_t^2 \to \infty$ in probability.

When θ_0 is on the boundary (zero coefficients):

- The asymptotic distribution cannot be normal

  When $\theta_0(i) = 0$, $\quad \sqrt{n}(\hat{\theta}(i) - \theta_0(i)) \geq 0$, a.s. for all n.

Technical assumptions

A1: $\theta_0 \in (\underline{\omega}, \overline{\omega}) \times [0, \overline{\theta}_2) \times \cdots \times [0, \overline{\theta}_{p+q+1}) \subset \Theta$, Θ compact.

A2: $\gamma(\mathbf{A_0}) < 0$ and $\sum_{j=1}^{p} \beta_j < 1, \quad \forall \theta \in \Theta$.

A3: η_t^2 is non-degenerate with $E\eta_t^2 = 1$ and $\kappa_\eta = E\eta_t^4 < \infty$.

A4: if $p > 0$, $\mathcal{A}_{\theta_0}(z)$ and $\mathcal{B}_{\theta_0}(z)$ have no common root, $\mathcal{A}_{\theta_0}(1) \neq 0$, and $\alpha_{0q} + \beta_{0p} \neq 0$.

Technical assumptions

- The matrix

$$J = E_{\theta_0}\left(\frac{1}{\sigma_t^4(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta'}\right)$$

may not exist without additional moment assumptions

A5: $E_{\theta_0}\epsilon_t^6 < \infty.$

or

A6: $\prod_{i=1}^{j_0}\alpha_{0i} > 0$ for $j_0 = \min\{j \mid \beta_{0,j} > 0\}.$

QMLE when the coefficient are allowed to be zero

$$\Lambda = \lim_{n\to\infty} \sqrt{n}(\Theta - \theta_0) = \Lambda_1 \times \cdots \times \Lambda_{p+q+1},$$

$$\Lambda_i = \mathbb{R} \quad \text{if} \quad \theta_{0i} \neq 0, \qquad \Lambda_i = [0, \infty) \quad \text{if} \quad \theta_{0i} = 0.$$

Theorem

Under the previous assumptions,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \quad \overset{d}{\to} \quad \lambda^{\Lambda} := \arg\inf_{\lambda \in \Lambda} \{\lambda - Z\}' J \{\lambda - Z\},$$

$$Z \sim \mathcal{N}\left(0, (\kappa_\eta - 1)J^{-1}\right),$$

[FZ, 2007]

Testing the nullity of GARCH coefficients

Motivations:

- Before proceeding to the estimation of a GARCH model, it is sensible to make sure that such a sophisticated model is justified.

- When a GARCH effect is present in the data, it is of interest to test if the orders of the fitted models can be reduced, by testing the nullity of the higher-lag ARCH or GARCH coefficient.

- Because the QMLE is positively constrained, its asymptotic distribution is not gaussian, and thus standard tests (such as the Wald or LR tests) based on the QMLE do not have the usual χ^2 asymptotic distribution.

Hypotheses

$$\theta_0 = (\theta_0^{(1)}, \theta_0^{(2)})', \quad \theta^{(i)} \in \mathbb{R}^{d_i}, \quad d_1 + d_2 = p + q + 1.$$

Null hypothesis:

H_0 : $\theta_0^{(2)} = 0$ i.e. $K\theta_0 = 0_{d_2 \times 1}$ with $K = \left(\, 0, \; I_{d_2} \, \right)$.

Maintained assumption:

H : $\theta_0^{(1)} > 0$ i.e. $\overline{K}\theta_0 > 0$ with $\overline{K} = \left(\, I_{d_1}, \; 0_{d_1 \times d_2} \, \right)$.

Local one-sided alternatives:

H_n : $\theta = \theta_0 + \frac{\tau}{\sqrt{n}}$, with $\theta_0^{(2)} = 0, \; \tau \in (0, +\infty)^{p+q+1}$.

Testing problems in which, under the null, the parameter is on the boundary of the maintained assumption:

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Testing when a parameter is on a boundary of the maintained hypothesis.
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Tests against one-sided alternatives:

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Wolak, F. A.
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A locally most mean powerful based score test for ARCH and GARCH regression disturbances.
Journal of Business and Economic Statistics 11, 17–27, 1993.

Usual forms of the Wald, Rao and QLR statistics

$$\begin{aligned}
\mathbf{W}_n &= \frac{n}{\hat{\kappa}_\eta - 1}\hat{\theta}_n^{(2)\prime}\left\{K\hat{J}_n^{-1}K'\right\}^{-1}\hat{\theta}_n^{(2)}, \\
\mathbf{R}_n &= \frac{n}{\hat{\kappa}_{\eta|2} - 1}\frac{\partial\tilde{\mathbf{l}}_n\left(\hat{\theta}_{n|2}\right)}{\partial\theta'}\hat{J}_{n|2}^{-1}\frac{\partial\tilde{\mathbf{l}}_n\left(\hat{\theta}_{n|2}\right)}{\partial\theta}, \\
\mathbf{L}_n &= n\left\{\tilde{\mathbf{l}}_n\left(\hat{\theta}_{n|2}\right) - \tilde{\mathbf{l}}_n\left(\hat{\theta}_n\right)\right\},
\end{aligned}$$

$\hat{\theta}_{n|2}$: constrained estimator of θ_0.

Standard (invalid) asymptotic critical regions at level α :

$$\{\mathbf{W}_n > \chi^2_{d_2}(1-\alpha)\}, \quad \{\mathbf{R}_n > \chi^2_{d_2}(1-\alpha)\}, \quad \{\mathbf{L}_n > \chi^2_{d_2}(1-\alpha)\}.$$

Asymptotic distributions of the statistics under the null

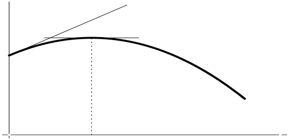
Under H_0 and the assumptions required for the asymptotic distribution of the QMLE

$$\begin{aligned}\mathbf{W}_n &\overset{\mathcal{L}}{\to} \mathbf{W} = \lambda^{\Lambda'}\Omega\lambda^{\Lambda},\\ \mathbf{R}_n &\overset{\mathcal{L}}{\to} \chi^2_{d_2},\\ \mathbf{L}_n &\overset{\mathcal{L}}{\to} \mathbf{L}\\ &= -\frac{1}{2}\left\{\inf_{K\lambda\geq 0}\|Z-\lambda\|_J^2 - \inf_{K\lambda=0}\|Z-\lambda\|_J^2\right\}.\end{aligned}$$

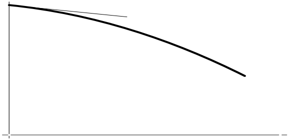
$$\Omega = K'\left\{(\kappa_\eta - 1)KJ^{-1}K'\right\}^{-1}K.$$

$\alpha \mapsto \log L_n(\hat{\omega}, \alpha)$ for an ARCH(1) with $\alpha_0 = 0$

$$\hat{\alpha}_n > 0 \quad \Longrightarrow \quad \mathbf{W}_n > 0, \mathbf{R}_n > 0, \mathbf{L}_n > 0$$



$$\hat{\alpha}_n = 0 \quad \Longrightarrow \quad \mathbf{W}_n = \mathbf{L}_n = 0, \mathbf{R}_n > 0$$



Power comparisons under fixed alternatives

In Bahadur's (1960) approach the efficiency of a test is measured by the rate of convergence of its p-value under a fixed alternative $H_1 : \theta_0^{(2)} > 0$.

Let $S_{\mathbf{W}}(t) = \mathbb{P}(\mathbf{W} > t)$, $S_{\mathbf{R}}(t) = \mathbb{P}(\mathbf{R} > t)$ where $\mathbf{R} \sim \chi^2_{d_2}$, and $S_{\mathbf{L}}(t) = \mathbb{P}(\mathbf{L} > t)$
(asymptotic survival functions of the statistics under H_0.)

Power comparisons under fixed alternatives

Proposition

Under $H_1 : \theta_0^{(2)} > 0$ and under **A1-A4**, the approximate Bahadur slope of the Wald test is

$$\lim_{n\to\infty} -\frac{2}{n} \log S_{\mathbf{W}}(\mathbf{W}_n) = \frac{1}{\kappa_\eta - 1} \theta_0^{(2)'} \left(KJ^{-1}K'\right)^{-1} \theta_0^{(2)}, \quad a.s.$$

Moreover, under regularity conditions, with $\theta_{0|2} = a.s. \lim \hat{\theta}_{n|2}$,

$$\lim_{n\to\infty} -\frac{2}{n} \log S_{\mathbf{R}}(\mathbf{R}_n) = \frac{1}{\kappa_{\eta|2} - 1} D'(\theta_{0|2}) K J_{0|2}^{-1} K' D(\theta_{0|2}),$$

$$\lim_{n\to\infty} -\frac{2}{n} \log S_{\mathbf{L}}(\mathbf{L}_n) = E_{\theta_0} \left(\log \frac{\sigma_t^2(\theta_{0|2})}{\sigma_t^2(\theta_0)} \right).$$

It follows that the Wald, score and QLR tests are consistent against H_1.

Distributions under local alternatives

$$H_n(\tau)\ :\quad \theta = \theta_0 + \frac{\tau}{\sqrt{n}} = \theta_n, \qquad K\theta_0 = 0 \text{ and } \tau \in [0, +\infty)^{p+q+1}.$$

Theorem

Under $H_n(\tau)$,

$$\begin{aligned}
\sqrt{n}(\hat{\theta}_n - \theta_n) &\overset{\mathcal{L}}{\to} \arg\inf_{\lambda\in\Lambda} \{\lambda - Z - \tau\}' J \{\lambda - Z - \tau\} - \tau, \\
&:= \lambda^{\Lambda}(\tau) - \tau \\
\mathbf{W}_n &\overset{\mathcal{L}}{\to} \mathbf{W}(\tau) = \lambda^{\Lambda}(\tau)'\Omega\lambda^{\Lambda}(\tau), \\
\mathbf{R}_n &\overset{\mathcal{L}}{\to} \chi^2_{d_2}\left\{\tau'\Omega\tau\right\}, \\
\mathbf{W}_n &\overset{o_P(1)}{=} \frac{2}{\hat{\kappa}_\eta - 1}\mathbf{L}_n.
\end{aligned}$$

$H_0 : \alpha_{0i} = 0$ (or $H_0 : \beta_{0j} = 0$)

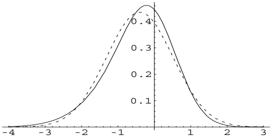
ex: GARCH$(p-1, q)$ vs GARCH(p, q).
Under H_0: $\theta_0 = (\theta_{01}, \theta_{02}, \dots, \theta_{0,p+q}, 0)$

$$\Lambda = \mathbb{R}^{p+q} \times [0, \infty), \qquad \gamma_i = \frac{E(Z_{p+q+1} Z_i)}{\mathrm{Var}(Z_{p+q+1})}$$

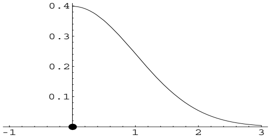
$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \lambda^{\Lambda} = \begin{pmatrix} Z_1 - \gamma_1 Z_{p+q+1}^- \\ \vdots \\ Z_{p+q} - \gamma_{p+q} Z_{p+q+1}^- \\ Z_{p+q+1}^+ \end{pmatrix}$$

Example: Noise estimated as an ARCH(1): $\theta_0 = (\omega_0, 0)'$

Asymptotic distribution of $\sqrt{n}(\hat{\omega}_n - \omega_0)$



Asymptotic distribution of $\sqrt{n}\hat{\alpha}_n$



$H_0 : \alpha_{0i} = 0$ (or $H_0 : \beta_{0j} = 0$)

Asymptotic distribution of the Wald and LR test statistics:

$$\mathbf{W} = \frac{2}{\kappa_\eta - 1}\mathbf{L} \rightsquigarrow \frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1.$$

The tests defined by the critical regions

$$\{\mathbf{W}_n > \chi^2_1(1-2\alpha)\} \qquad \{\frac{2}{\hat{\kappa}_\eta - 1}\mathbf{L}_n > \chi^2_1(1-2\alpha)\}$$

have asymptotic level α (for $\alpha \leq 1/2$).

The standard test $\{\mathbf{W}_n > \chi^2_1(1-\alpha)\}$ has asymptotic level $\alpha/2$.

Asymptotic behaviour of the standard tests

Table: Asymptotic levels of the standard Wald and QLR tests of nominal level 5%.

Kurtosis of η	2	3	4	5	6	7	8	9	10
Standard Wald	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
Standard QLR	0.3	2.5	5.5	8.3	10.8	12.9	14.7	16.4	17.8

Comparison of the modified tests under local alternatives

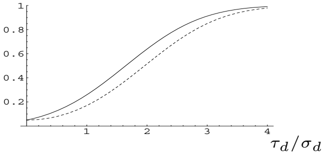
Proposition

Under $H_n(\tau)$: $\theta = \theta_0 + \frac{\tau}{\sqrt{n}}$, $\tau > 0$, and $d_2 = 1$,

$$\lim_{n\to\infty} P\left\{\mathbf{W}_n > \chi_1^2(1-2\alpha)\right\} > \lim_{n\to\infty} P\left\{\mathbf{R}_n > \chi_1^2(1-\alpha)\right\}.$$

Local asymptotic powers ($d_2 = 1$)

Modified Wald test (full line)
Score test (dashed line)



Optimality of the modified Wald test ($d_2 = 1$)

LAN property for GARCH models (Drost and Klaassen (1997), Ling and McAleer (2003))

Assume η_t has density f with $\int \{1 + yf'(y)/f(y)\}^2 f(y)dy < \infty$.

Corollary

The modified Wald test is asymptotically optimal iff the density f of η_t is of the form

$$f(y) = \frac{a^a}{\Gamma(a)} \exp(-ay^2)|y|^{2a-1}, \qquad a > 0.$$

Testing conditional homoskedasticity versus ARCH(q): $H_0 : \theta_0 = (\omega_0, 0, \dots, 0)$

$$\Lambda = \mathbb{R} \times [0, \infty)^q.$$

We have, with $e = (1, \dots, 1)'$

$$Z \sim \mathcal{N}\left\{0, (\kappa_\eta - 1)J^{-1} = \begin{pmatrix} (\kappa_\eta + 1)\omega_0^2 & -\omega_0 e' \\ -\omega_0 e & I_q \end{pmatrix}\right\}.$$

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \lambda^\Lambda = \begin{pmatrix} Z_1 + \omega_0(Z_2^- + \cdots + Z_{q+1}^-) \\ Z_2^+ \\ \vdots \\ Z_{q+1}^+ \end{pmatrix}.$$

Testing conditional homoskedasticity versus ARCH(q): $H_0 : \alpha_{01} = \cdots = \alpha_{0q} = 0$

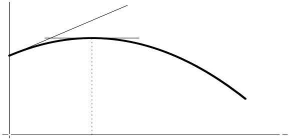
Some simple statistics:

- As noted by Engle (1982), the score test is very simple to compute:
$$\mathbf{R}_n = nR^2,$$
where R^2 is the determination coefficient in the regression of ϵ_t^2 on a constant and $\epsilon_{t-1}^2, \ldots, \epsilon_{t-q}^2$.
- An asymptotically equivalent version is
$$\mathbf{R}_n^* = n \sum_{i=1}^{q} \hat{\rho}_{\epsilon^2}^2(i),$$
where $\hat{\rho}_{\epsilon^2}(i)$ is an estimator of the i-th autocorrelation of (ϵ_t^2).

- The Wald statistic also has a simple version:

$$\mathbf{W}_n^* = n \sum_{i=1}^{q} \hat{\alpha}_i^2.$$

- Lee and King (1993) proposed a test which exploits the one-sided nature of the ARCH alternative.



$$\mathbf{LK}_n = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} \sqrt{n}\hat{\rho}_{\epsilon^2}(i).$$

Asymptotic null distributions

Proposition

Under H_0 and **A3** (η_t^2 non-degenerate, $E\eta_t^2 = 1$, $E\eta_t^4 < \infty$),

$$\begin{aligned} \mathbf{W}_n^* &\xrightarrow{d} \frac{1}{2^q}\delta_0 + \sum_{i=1}^{q} \binom{q}{i} \frac{1}{2^q}\chi_i^2, \\ \mathbf{R}_n^* &\xrightarrow{d} \chi_q^2, \\ \mathbf{LK}_n &\xrightarrow{d} \mathcal{N}(0,1). \end{aligned}$$

Power comparisons under fixed alternatives

Asymptotic relative efficiencies (ARE) are defined by the ratios of the approximate Bahadur slopes.

Proposition

Let (ϵ_t) be a strictly stationary and nonanticipative solution of the ARCH(q) model with $E(\epsilon_t^4) < \infty$ and $\sum_{i=1}^q \alpha_{0i} > 0$. Then,

$$\begin{aligned}
\mathbf{ARE}(\mathbf{R}^*/\mathbf{LK}) &= \frac{q\sum_{i=1}^q \rho_{\epsilon^2}^2(i)}{\left\{\sum_{i=1}^q \rho_{\epsilon^2}(i)\right\}^2} \geq 1,\\
\mathbf{ARE}(\mathbf{R}^*/\mathbf{W}^*) &= \frac{\sum_{i=1}^q \rho_{\epsilon^2}^2(i)}{\sum_{i=1}^q \alpha_{0i}^2} \geq 1,\\
\mathbf{ARE}(\mathbf{R}/\mathbf{W}^*) &= \frac{\kappa_\epsilon - \kappa_\eta}{\kappa_\eta(\kappa_\epsilon - 1)\sum_{i=1}^q \alpha_{0i}^2} \geq 1,
\end{aligned}$$

with equalities when $q = 1$.

Efficiency rankings under fixed alternatives

ARCH(1) alternative:

$$\mathbf{W} \prec \mathbf{L} \prec \mathbf{R} \sim \mathbf{R}^* \sim \mathbf{W}^* \sim \mathbf{LK}$$

ARCH(2) alternative:

$$\mathbf{W} \prec \mathbf{L} \prec \mathbf{W}^* \prec \mathbf{R} \prec \mathbf{R}^*.$$

The LK cannot be ranked in general: it can have the lowest or the highest asymptotic efficiency depending on the parameter values.

Local asymptotic powers ($d_2 = q$)

Under the local alternatives $H_n(\tau)$, $\tau > 0$, the local asymptotic powers are given by

$$\begin{aligned}
\lim_{n\to\infty} P\left\{\mathbf{W}_n > c_\alpha^{\mathbf{W}}\right\} &= P\left\{\sum_{i=1}^{q}(U_i+\tau_i)^2\mathbf{1}_{\{U_i+\tau_i>0\}} > c_\alpha^{\mathbf{W}}\right\}\\
\lim_{n\to\infty} P\left\{\mathbf{R}_n > c_\alpha^{\mathbf{R}}\right\} &= P\left\{\chi_q^2\left(\sum_{i=1}^{q}\tau_i^2\right) > c_\alpha^{\mathbf{R}}\right\}\\
\lim_{n\to\infty} P\left\{\mathbf{LK}_n > c_\alpha\right\} &= 1-\Phi\left(c_\alpha - \frac{\sum_{i=1}^{q}\tau_i}{\sqrt{q}}\right),
\end{aligned}$$

where $U = (U_1, \dots, U_q)' \sim \mathcal{N}(0, I_q)$.

The LK test is locally asymptotically optimal in the direction $\tau_1 = \cdots = \tau_q$ when $f(y) = \frac{a^a}{\Gamma(a)} \exp(-ay^2)|y|^{2a-1}, \quad a > 0.$

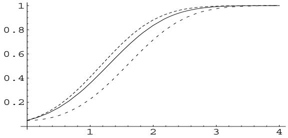
Moreover, it is locally asymptotically "most stringent somewhere most powerful".

(see Akharif and Hallin (2003) for the concept of MSSMP).

Local asymptotic powers ($d_2 = 2$)

Wald test (full line), score test (dashed line), Lee-King test (dotted line)

$\alpha_1 = \alpha_2 = \tau/\sqrt{n}$



$\alpha_1 = \tau/\sqrt{n},\ \alpha_2 = 0$ (or $\alpha_1 = 0,\ \alpha_2 = \tau/\sqrt{n}$)

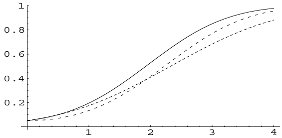


Table: p-values for tests of the null hypothesis of a $GARCH(1,1)$ model for daily stock market returns.

Index	alternative								
	GARCH(1,2)			GARCH(1,4)			GARCH(2,1)		
	$\mathbf{W}_n$	$\mathbf{R}_n$	$\mathbf{L}_n$	$\mathbf{W}_n$	$\mathbf{R}_n$	$\mathbf{L}_n$	$\mathbf{W}_n$	$\mathbf{R}_n$	$\mathbf{L}_n$
CAC	**0.018**	0.069	**0.028**	**0.006**	**0.000**	**0.003**	0.500	0.457	0.500
DAX	**0.004**	**0.002**	**0.005**	**0.002**	**0.000**	**0.001**	0.335	**0.022**	0.119
DJA	0.318	0.653	0.323	0.471	0.379	0.475	0.500	0.407	0.500
DJI	0.089	0.203	0.098	0.168	0.094	0.179	0.500	**0.024**	0.500
DJT	0.500	0.743	0.500	0.649	**0.004**	0.649	0.364	0.229	0.251
DJU	0.500	**0.000**	0.500	0.648	**0.000**	0.648	**0.004**	**0.000**	**0.002**
FTSE	0.131	0.210	0.119	0.158	0.357	0.143	0.414	0.678	0.380
Nasdaq	0.053	0.263	0.092	0.067	**0.002**	0.123	0.500	0.222	0.500
Nikkei	**0.010**	**0.003**	**0.008**	0.090	0.479	0.143	0.201	**0.000**	**0.015**
SP 500	0.116	0.190	0.107	0.075	**0.029**	0.055	0.500	0.178	0.500

Conclusions

- Caution is needed in the use of standard statistics for testing the nullity of coefficients in GARCH models, because the null hypothesis puts the parameter at the boundary of the parameter space.
- The asymptotic sizes of the *standard* Wald and QLR tests can be very different from the *nominal* levels based on (invalid) χ^2 distributions.
- The modified Wald and QLR tests remain equivalent under the null and local alternatives.
- The usual Rao test remains valid for testing a value on the boundary, but looses its local optimality properties.

- For testing the nullity of one coefficient the modified Wald and QLR tests are locally asymptotically optimal for a certain class of densities.
- For testing conditional homoscedasticity:
  the one-sided Lee-King test has optimality properties but only for alternatives in certain directions.
  The modified Wald test

$$\left\{ n \sum_{i=1}^{q} \hat{\alpha}_i^2 > c_{q,\alpha} \right\}, \quad P\left(\frac{1}{2^q}\delta_0 + \sum_{i=1}^{q} \binom{q}{i} \frac{1}{2^q}\chi_i^2 > c_{q,\alpha} \right) = \alpha,$$

  can be recommended: from both local and non local points of view, theoretical and numerical results suggest that it is always close to the optimum.
- The GARCH(1,1) is certainly over-represented in financial studies.